

The mean square of the product of the Riemann zeta-function with Dirichlet polynomials

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Abstract. Improving earlier work of Balasubramanian, Conrey and Heath-Brown [1], we obtain an asymptotic formula for the mean-square of the Riemann zeta-function times an arbitrary Dirichlet polynomial of length $T^{\frac{1}{2}+\delta}$, with $\delta = 0.01515\dots$. As an application we obtain an upper bound of the correct order of magnitude for the third moment of the Riemann zeta-function. We also refine previous work of Deshouillers and Iwaniec [8], obtaining asymptotic estimates in place of bounds. Using the work of Watt [19], we compute the mean-square of the Riemann zeta-function times a Dirichlet polynomial of length going up to $T^{\frac{3}{4}}$ provided that the Dirichlet polynomial assumes a special shape. Finally, we exhibit a conjectural estimate for trilinear sums of Kloosterman fractions which implies the Lindelöf Hypothesis.

1. Introduction

We are interested in the mean-square of the product of the Riemann zeta-function $\zeta(s)$ with an arbitrary Dirichlet polynomial $A(s)$. More precisely, we would like to understand

$$(1.1) \quad I = \int_{\mathbb{R}} |\zeta(\tfrac{1}{2} + it)|^2 \cdot |A(\tfrac{1}{2} + it)|^2 \phi\left(\frac{t}{T}\right) dt$$

with $\phi(x)$ a smooth function supported in $[1, 2]$ and

$$A(s) := \sum_{n \leq T^\theta} \frac{a_n}{n^s}, \quad a_n \ll n^\varepsilon, \quad \theta < 1.$$

Asymptotic estimates for I have been used consistently to understand the distribution of values of L -functions, the location of their zeros, and upper and lower bounds for the size of L -functions. See, for example, [4, 5, 16, 18].

It is crucially important to allow θ to be as large as possible. For example, if we could take $\theta = 1 - \varepsilon$ in (1.1), then the Lindelöf Hypothesis would follow.

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Balasubramanian, Conrey and Heath-Brown obtained an asymptotic formula for I when $\theta < \frac{1}{2}$. For $\theta < \frac{1}{2}$ and $\phi(t)$ the indicator function of the interval $[1, 2]$, they show that

$$(1.2) \quad I = T \sum_{d,e \leq T^\theta} \frac{a_d \overline{a_e}}{[d,e]} \cdot \left(\log \left(\frac{T(d,e)^2}{2\pi de} \right) + 2\gamma + \log 4 - 1 \right) + o(T).$$

When $A(s)$ is a mollifier, they show that one can go further and take

$$\theta < \frac{1}{2} + \frac{1}{34} = 0.529411 \dots$$

Their motivation was to understand the location of the zeros of the Riemann zeta-function. Specifically they deduce that at least 38% of the complex zeros of $\zeta(s)$ are on the critical line $\Re(s) = \frac{1}{2}$. Improvements on the admissible length of $A(s)$ will lead to a further understanding of the zeros of $\zeta(s)$ on the critical line. (See also [4].)

In complete generality, formula (1.2) fails when $\theta > 1$. Balasubramanian, Conrey and Heath-Brown conjecture that it remains true provided that $\theta < 1$. This is known as the $\theta = 1$ conjecture. An important change occurs at $\theta = \frac{1}{2}$. When $\theta < \frac{1}{2}$, only the diagonal terms (in the sense of Section 3.1 below) contribute to I , while for $\theta > \frac{1}{2}$ there is also a contribution from the non-diagonal terms which seems difficult to manage given the generality of the Dirichlet polynomial $A(s)$. The main result of our paper consists in breaking the $\frac{1}{2}$ barrier for an arbitrary Dirichlet polynomial. In fact, we prove (1.2) for $\theta < \frac{17}{33} = \frac{1}{2} + \delta$ with $\delta = \frac{1}{66} \approx 0.01515 \dots$

Theorem 1. *Let I and $A(s)$ be as above. If $\theta < \frac{1}{2} + \delta$, with $\delta = \frac{1}{66}$, then*

$$I = \sum_{d,e \leq T^\theta} \frac{a_d \overline{a_e}}{[d,e]} \cdot \int_{\mathbb{R}} \left(\log \left(\frac{t(d,e)^2}{2\pi de} \right) + 2\gamma \right) \phi \left(\frac{t}{T} \right) dt + O(T^{\frac{3}{20}+\varepsilon} N^{\frac{33}{20}} + T^{\frac{1}{3}+\varepsilon}),$$

where $N := T^\theta$.

We notice that the off-diagonal terms contribute to the main term roughly those d and e for which the logarithm in the above expression is negative.

Our main tool in the proof of Theorem 1 is an estimate for trilinear forms of Kloosterman fractions, which will appear in [2]. This estimate improves a result of Duke, Friedlander, Iwaniec in [9], dealing with bilinear sums. The use of Theorem 2 in their paper is also enough to break the $\frac{1}{2}$ barrier, though with the smaller constant $\delta = \frac{1}{190} \approx 0.00526$ in Theorem 1.

If we assume a general estimate for trilinear forms of Kloosterman fractions, such as

$$(1.3) \quad \begin{aligned} S_{A,M,N} &:= \sum_a \sum_{(m,n)=1} v_a \alpha_m \beta_n e \left(\frac{a\overline{m}}{n} \right) \\ &\ll_{\varepsilon} \|\alpha\| \|\beta\| \|\nu\| (M+N)^{\frac{1}{2}+r+\varepsilon} A^t \\ &\quad + \|\nu\| A^{\frac{1}{2}} (\|\alpha\|_{\infty} \|\beta\| N^{\frac{1}{2}+\varepsilon} + \|\alpha\| \|\beta\|_{\infty} M^{\frac{1}{2}+\varepsilon}), \end{aligned}$$

where $M \leq m < 2M$, $N \leq n < 2N$, $A \leq a < 2A$, $A \ll (NM)^{\frac{1-r}{1+2t}+\varepsilon}$, and $\|\cdot\|$ and $\|\cdot\|_{\infty}$ denote the L_2 and L_{∞} norms respectively, then the statement of Theorem 1 can be replaced as follows.

Theorem 2. Suppose that (1.3) is true for some $r, t \geq 0$. Then

$$I = \sum_{d,e \leq T^\theta} \frac{a_d \overline{a_e}}{[d,e]} \cdot \int_{\mathbb{R}} \left(\log \left(\frac{t(d,e)^2}{2\pi de} \right) + 2\gamma \right) \phi \left(\frac{t}{T} \right) dt + O(T^{\frac{1}{2}-t+\varepsilon} N^{\frac{1}{2}+r+2t} + T^{\frac{1}{3}+\varepsilon})$$

for $\theta < \frac{1}{2} + \frac{\frac{1}{2}-r}{1+2(r+2t)}$ and where $N := T^\theta$.

The estimate of Duke, Friedlander, Iwaniec implies (1.3) with $r = \frac{23}{48}$ and $t = \frac{1}{2}$, while the estimate of Bettin and Chandee allows us to take $r = \frac{9}{20}$ and $t = \frac{7}{20}$. We conjecture that bound (1.3) holds true for all $r, t \geq 0$.

Conjecture 1. Let $A \ll (NM)^{\frac{1}{2}+\varepsilon}$. Then

$$(1.4) \quad S_{A,M,N} \ll \|\alpha\| \|\beta\| \|\nu\| (M+N)^{\frac{1}{2}+\varepsilon} + \|\nu\| A^{\frac{1}{2}} (\|\alpha\|_\infty \|\beta\| N^{\frac{1}{2}+\varepsilon} + \|\alpha\| \|\beta\|_\infty M^{\frac{1}{2}+\varepsilon}).$$

This conjecture essentially states that we expect square-root cancellation in the shortest two sums, as long as the total saving does not exceed M or N . In the Appendix we show that this is best possible, up to ε -powers.

Using the estimate (1.4) and Theorem 2, we obtain an asymptotic formula for I valid for any $\theta < 1$, and this implies the Lindelöf Hypothesis. We state this as a corollary below.

Corollary 1. Suppose that Conjecture 1 holds. Then the Lindelöf Hypothesis is true.

Conjecture 1 appears to be strictly stronger than the Lindelöf Hypothesis. Indeed, Conjecture 1 implies (1.2) with $\theta < 1$, while the Lindelöf Hypothesis only gives the cruder bound

$$I \ll T^{1+\varepsilon} \sum_{n \leq T^\theta} \frac{|a_n|^2}{n}.$$

The proof of Theorem 2, on which Corollary 1 depends, is the same as that of Theorem 1 except that we use (1.3) instead of Proposition 1. The modification will be discussed at the end of Section 3.

Duke, Friedlander and Iwaniec apply their estimate to obtain bounds for the twisted second moment of a Dirichlet L -function [10]. They show that

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2}, \chi)|^2 \cdot |D(\tfrac{1}{2}, \chi)|^2 \ll q^{1+\varepsilon}$$

for Dirichlet polynomials $D(s, \chi)$ with coefficients $a_n \ll n^\varepsilon$ and of length $q^{\frac{1}{2}+\delta'}$ with some $\delta' > 0$. Our proof of Theorem 1 would not extend to give an asymptotic formula in this case, and additional input is needed.

As an application of Theorem 1 we obtain an upper bound of the correct order of magnitude for the third moment of the Riemann zeta-function.

Corollary 2. We have,

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^3 dt \ll T(\log T)^{\frac{9}{4}}.$$

We further indicate in Section 6.1 how to refine this result to obtain correct upper bounds for the $2k$ -th moment, when k has the form $k = 1 + \frac{1}{n}$. Previously, Corollary 2 was known only on the assumption of the Riemann Hypothesis [13]. The only sharp unconditional upper bounds that were previously known are for the classic cases $k = 0, 1, 2$ and for $k = \frac{1}{n}$, due to Heath-Brown [13].

With further applications in mind we investigate how much θ can be increased when the Dirichlet polynomial $A(s)$ is specialized.

1.1. Products of two Dirichlet polynomials. When $A(s)$ can be written as a product of two Dirichlet polynomials $B(s)C(s)$, one can appeal to stronger estimates for sums of Kloosterman sums due to Deshouillers and Iwaniec. In [8], Deshouillers and Iwaniec consider the product of $\zeta(s)$ with two Dirichlet polynomials,

$$(1.5) \quad J = \int_{\mathbb{R}} |\zeta(\tfrac{1}{2} + it)|^2 \cdot |A(\tfrac{1}{2} + it)|^2 \cdot |B(\tfrac{1}{2} + it)|^2 dt$$

with

$$(1.6) \quad A(s) := \sum_{n \leq N} \frac{\alpha_n}{n^s}, \quad B(s) := \sum_{k \leq K} \frac{\beta_k}{k^s}, \quad \text{where } \alpha_n \ll n^\varepsilon, \beta_k \ll k^\varepsilon.$$

They show that if $N \geq K$, then

$$J \ll T^\varepsilon \cdot (T + T^{\frac{1}{2}} N^{\frac{3}{4}} K + T^{\frac{1}{2}} N K^{\frac{1}{2}} + N^{\frac{7}{4}} K^{\frac{3}{2}}).$$

Their proof depends on estimates for incomplete Kloosterman sums as developed in [8]. Proceeding similarly as in the proof of Theorem 1, and using Deshouillers and Iwaniec's estimate, we refine their bound to an asymptotic estimate.

Theorem 3. *Let $J, A(s)$ and $B(s)$ be as defined in (1.5) and (1.6), and let $N \geq K$. Then,*

$$J = \sum_{d,e \leq NK} \frac{a_d \overline{a_e}}{[d,e]} \cdot \int_{\mathbb{R}} \left(\log \left(\frac{t(d,e)^2}{2\pi d e} \right) + 2\gamma \right) \phi \left(\frac{t}{T} \right) dt \\ + O(T^\varepsilon \cdot (T^{\frac{1}{2}} N^{\frac{3}{4}} K + T^{\frac{1}{2}} N K^{\frac{1}{2}} + N^{\frac{7}{4}} K^{\frac{3}{2}})),$$

with

$$a_d := \sum_{nk=d} \alpha_n \beta_k.$$

When the length of N and K is chosen suitably, Theorem 3 allows us to take $\theta < \frac{1}{2} + \frac{1}{10}$.

1.2. Specializing one of the Dirichlet polynomials. A specific case of interest is the product $A(s)B(s)$ with $A(s)$ of length $N = \sqrt{T}$ and smooth coefficients, and $B(s)$ arbitrary and as long as possible. One can think of such estimates as estimates for the twisted fourth moment of the Riemann zeta-function. In this case, we can go further by combining the trilinear sums estimate used to prove Theorem 1 with Watt's strengthening [19] of the groundbreaking work of Deshouillers-Iwaniec on estimates for sums of Kloosterman sums [7].

Theorem 4. *Let $J, A(s)$ and $B(s)$ be as defined in (1.5) and (1.6). Let $N \ll T^{\frac{1}{2}+\varepsilon}$ for all $\varepsilon > 0$ and assume that $\alpha_n = \psi(n)$ with $\psi(x)$ a smooth function such that $\psi^{(j)}(x) \ll_j x^{-j}$*

for all $j > 0$. Let $K \ll T^{\frac{1}{4}}$ and $\beta_k \ll k^\varepsilon$ for all $\varepsilon > 0$. Moreover, assume α_n is supported on $[NT^{-\xi_1}, 2N]$ and β_k is supported on $[KT^{-\xi_2}, 2K]$, where $0 \leq \xi_1 \leq \frac{1}{5}$, $0 \leq \xi_2 \leq \frac{1}{16}$. Then,

$$J = \sum_{d,e \leq NK} \frac{a_d \overline{a_e}}{[d,e]} \cdot \int_{\mathbb{R}} \left(\log \left(\frac{t(d,e)^2}{2\pi de} \right) + 2\gamma \right) \phi \left(\frac{t}{T} \right) dt \\ + O(T^{\frac{1}{2}+\varepsilon} K^2 + KN^{\frac{3}{4}} T^{\frac{3}{8}+\varepsilon} + T^{\frac{39}{40}+\frac{1}{8}\xi_1+\frac{2}{5}\xi_2+\varepsilon}),$$

where

$$a_d = \sum_{nk=d} \alpha_n \beta_k.$$

Remark. Theorem 4 yields an asymptotic formula for $5\xi_1 + 16\xi_2 < 1$ (and $N \ll T^{\frac{1}{2}}$, $K \ll T^{\frac{1}{2}-\varepsilon}$). We remark that this range could be enlarged with a little more work.

We notice that Theorem 4 allows us to take $\theta < \frac{3}{4}$ for Dirichlet polynomials of the form $A(s)B(s)$ with $A(s)$ pretending to be $\zeta(s)$ and $B(s)$ of length up to $T^{\frac{1}{4}-\varepsilon}$. Thus, following the work of Radziwiłł [17], Theorem 4 could be applied to give a sharp upper bound for the $2k$ -th moment of the Riemann zeta-function for $2k < 5$, conditionally on the Riemann Hypothesis (however, we remark that this has recently been proven for all $k \geq 0$ by Harper [12]). It would be interesting to investigate if Theorem 4 has other applications, for example to the study of large gaps between the zeros of the Riemann zeta-function (see [3]).

Theorem 4 refines upon Watt's result, who uses his Kloosterman sum estimate to give (essentially) an upper bound of the form

$$J \ll T^{1+\varepsilon} + T^{\frac{1}{2}+\varepsilon} K^2,$$

for a_n and b_n supported on dyadic intervals. Theorem 4 should also be compared with the asymptotic formula for the twisted fourth moment of Hughes and Young [14]. Their result allows to get an asymptotic formula for the second moment of $\zeta^2(s)B(s)$ with $B(s)$ of length up to $T^{\frac{1}{11}-\varepsilon}$.

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2. Estimates for sums of Kloosterman sums

Remark. Throughout the paper, we use the common convention in analytic number theory that ε denotes an arbitrarily small positive quantity that may vary from line to line.

In this section, we collect the estimates for sums of Kloosterman sums that will be used to prove the theorems.

The following proposition is from [2], and we will use it when dealing with the contribution of the off-diagonal terms in Theorems 1 and 4.

Proposition 1. *Let α_m, β_n, ν_a be complex numbers, where $M \leq m < 2M$, $N \leq n < 2N$ and $A \leq a < 2A$. Then for any $\varepsilon > 0$, we have*

$$(2.1) \quad \sum_a \sum_{(m,n)=1} \sum \nu_a \alpha_m \beta_n e\left(\frac{a\bar{m}}{n}\right) \\ \ll_{\varepsilon} \|\alpha\| \|\beta\| \|v\| \left(1 + \frac{A}{MN}\right)^{\frac{1}{2}} \\ \times ((AMN)^{\frac{7}{20}+\varepsilon} (M+N)^{\frac{1}{4}} + (AMN)^{\frac{3}{8}+\varepsilon} (AN+AM)^{\frac{1}{8}}),$$

where $\|\cdot\|$ denotes the L_2 norm.

The off-diagonal terms in Theorem 3 will be estimated using the following bound, due to Deshouillers and Iwaniec [8].

Proposition 2 (Deshouillers, Iwaniec). *Let $L, J, U, V \geq 1$ and $|c(u, v)| \leq 1$. We have*

$$\sum_{1 \leq \ell \leq L} \sum_{\substack{1 \leq j \leq J \\ (\ell, \varrho j)=1}} \left| \sum_{1 \leq u \leq U} \sum_{\substack{1 \leq v \leq V \\ (v, \ell)=1}} c(u, v) e\left(u \frac{\overline{\varrho v j}}{\ell}\right) \right| \\ \ll (LJUV)^{\frac{1}{2}+\varepsilon} \{(LJ)^{\frac{1}{2}} + (U+V)^{\frac{1}{4}} [LJ(U+\varrho V)(L+\varrho V^2) + \varrho UV^2 J^2]^{\frac{1}{4}}\}.$$

Finally, to estimate the off-diagonal terms in Theorem 4, we will use the following proposition, which can be derived easily from Watt [19, Proposition 4.1].

Proposition 3 (Watt). *Let $H, C, P, V, R, S \geq 1$ and $\delta \leq 1$. Assume that for some $\varepsilon > 0$ we have*

$$(2.2) \quad X := \left(\frac{RVSP}{HC}\right)^{\frac{1}{2}} \gg (RSPV)^{\varepsilon}, \quad (RS)^2 \geq \max\left(H^2 C, \frac{SP}{V} (RSPV)^{\varepsilon}\right).$$

Moreover, assume that $\alpha(x), \beta(x)$ are complex valued smooth functions, supported on the intervals $[\frac{1}{2}, H]$ and $[\frac{1}{2}, C]$ respectively, such that

$$\alpha^{(j)}(x), \beta^{(j)}(x) \ll_j (\delta x)^{-j}$$

for all $j \geq 0$. Assume a_r, b_s are sequences of complex numbers supported on $[\frac{R}{2}, R]$, $[\frac{S}{2}, S]$ respectively and are such that $a_r \ll r^{\varepsilon}$, $b_s \ll s^{\varepsilon}$. Finally, assume that for all $i, j \geq 0$,

$$\frac{d^{i+j}}{dy^i dx^j} \gamma_{r,s}(x, y) \ll_{i,j} x^{-i} y^{-j},$$

where for all r and s , $\gamma_{r,s}(x, y)$ is supported on $[\frac{V}{2}, V] \times [\frac{P}{2}, P]$. Then

$$(2.3) \quad \sum_{s \sim S} \sum_{r \sim R} \sum_{\substack{h, c, p, v \\ (rv, sp)=1}} \sum \sum \alpha(h) \beta(c) \gamma_{r,s}(v, p) a_r b_s e\left(\pm \frac{hc\bar{r}v}{sp}\right) \\ \ll \delta^{-\frac{7}{2}} HCR(V+SX) \left(1 + \frac{HC}{RS}\right)^{\frac{1}{2}} \left(1 + \frac{P}{VR}\right)^{\frac{1}{2}} \\ \times \left(1 + \frac{H^2 CPX^2}{VS^3 R^4}\right)^{\frac{1}{4}} (HCRVPS)^{20\varepsilon}.$$

Proof. Firstly, using smooth partitions of unity, we can assume that $\alpha(x), \beta(x)$ are supported on $[\frac{H}{2}, H]$ and $[\frac{C}{2}, C]$, since the bound (2.3) is weaker (and the conditions (2.2) stricter) for larger values of H and C . Moreover, by dividing by R^ε and S^ε if necessary, we can assume $a_r, b_s \ll 1$.

By Poisson's formula,

$$\begin{aligned} \sum_{(v,sp)=1} \gamma(v,p) e\left(\pm \frac{hc\bar{r}v}{sp}\right) &= \sum_{u \pmod{sp}}^* e\left(\pm \frac{hc\bar{r}u}{sp}\right) \sum_{v \equiv u \pmod{sp}} \gamma(v,p) \\ &= \sum_{u \pmod{sp}}^* e\left(\pm \frac{hc\bar{r}u}{sp}\right) \frac{1}{sp} \sum_{\ell} e\left(-\frac{\ell u}{sp}\right) \int_{\mathbb{R}} \gamma(y,p) e\left(\frac{\ell y}{sp}\right) dy \\ &= \sum_{\ell} S(hc\bar{r}, \mp \ell, sp) \int_{\mathbb{R}} \gamma(y, sp, p) e(\ell y) dy. \end{aligned}$$

If $\ell = 0$, the Kloosterman sum reduces to a Ramanujan sum, and one has

$$S(hc\bar{r}, \mp \ell, sp) \ll (hc, sp).$$

Thus, the contribution to (2.3) coming from the terms $\ell = 0$ is bounded by

$$\sum_{s \sim S} \sum_{r \sim R} \sum_{\substack{h,c,p \\ (r,sp)=1}} \alpha(h) \beta(c) a_r b_s \frac{(hc, sp)}{sp} V \ll RVHC(HCP)^\varepsilon.$$

Also, integrating by parts repeatedly, we see that the terms with $\ell \geq \frac{SP}{V} (RSPV)^\varepsilon$ give a negligible contribution. For the remaining terms, we introduce a smooth partition of unity

$$1 = \sum_L' \theta_L(x) \quad \text{for all } x \geq 1,$$

where $\theta_L(x)$ is supported in $[\frac{L}{2}, 3L]$ (with $L \ll \frac{SP}{V} (RSPV)^\varepsilon$), satisfies $\theta_L(x)^j \ll_j L^{-j}$ for all $j \geq 0$, and is such that

$$\sum_{L \leq X}' 1 \ll \log(2+X) \quad \text{for all } X \geq 1.$$

Thus, we need to bound

$$\begin{aligned} &\sum_{s \sim S} \sum_{r \sim R} \sum_{\substack{h,c,p \\ (r,sp)=1}} \sum_{0 < |\ell| < \frac{SP}{V} (RSPV)^\varepsilon} \alpha(h) \beta(c) a_r b_s S(hc\bar{r}, \mp \ell, sp) \int_{\mathbb{R}} \gamma(y, sp, p) e(\ell y) dy \\ &= \sum_L' \int_{y \sim \frac{V}{SP}} \sum_{s \sim S} \sum_{r \sim R} \sum_{\substack{h,c,p \\ (r,sp)=1}} \sum_{\ell \sim L} \alpha(h) \beta(c) \omega(\ell, y) a_r b_s S(hc\bar{r}, \mp \ell, sp) f_s(p, y) dy, \end{aligned}$$

where $f_s(p, y) := \gamma(y, sp, p)$, $\omega(\ell, y) = \theta(\ell) e(\ell y)$, and \sum_L' denotes the sum over the partitions of unity. We remark that for $y \sim \frac{V}{SP}$, we have

$$\frac{d^j}{dp^j} f_s(p, y) \ll p^{-j},$$

and that

$$\frac{d^j}{d\ell^j} \omega(\ell, y) \ll \left(L^{-1} + \frac{V}{SP} \right)^j \ll L^{-j} (RSPV)^{j\varepsilon}.$$

By Watt [19, Proposition 4.1], the sums inside the integral are bounded by

$$\delta^{-\frac{7}{2}} (RVLP)^{\frac{7}{2}\varepsilon} HCL(RSX)^{1+\varepsilon} \left(1 + \frac{HC}{RS} \right)^{\frac{1}{2}} \left(1 + \frac{L}{RS} \right)^{\frac{1}{2}} \left(1 + \frac{H^2 CLX^2}{(RS)^4} \right)^{\frac{1}{4}},$$

and summing over L and integrating over y completes the proof of the proposition. \square

3. The proof of Theorem 1

We start by expressing $|\zeta(\frac{1}{2} + it)|^2$ as a sum of length approximately $T^{1+\varepsilon}$. Let $G(w)$ be an entire function with rapid decay along vertical lines, that is $G(x + iy) \ll y^{-A}$ for any fixed x and $A > 0$. Suppose $G(-w) = G(w)$, $G(0) = 1$, $G(\frac{1}{2}) = 0$. We will use the following form of the approximate functional equation for $|\zeta(s)|^2$.

Lemma 1 (Approximate functional equation). *For $T < t < 2T$, we have*

$$|\zeta(\tfrac{1}{2} + it)|^2 = 2 \sum_{m_1, m_2} \frac{1}{(m_1 m_2)^{\frac{1}{2}}} \left(\frac{m_1}{m_2} \right)^{it} W\left(\frac{2\pi m_1 m_2}{t} \right) + O(T^{-\frac{2}{3}}),$$

where

$$W(x) := \frac{1}{2\pi i} \int_{(2)} x^{-w} G(w) \frac{dw}{w},$$

and where we use the notation $\int_{(c)}$ to mean an integration up the vertical line from $c - i\infty$ to $c + i\infty$.

The proof of the lemma can be found in [15, Lemma 3].

Remark. Notice that $W^{(\ell)}(x) \ll_{\ell, A} \min(1, x^{-A})$ for $x > 0$ and all $\ell \in \mathbb{N}$.

The error term in Lemma 1 produces an error term bounded by $T^{\frac{1}{3}+\varepsilon}$, and thus

$$\begin{aligned} I &= 2 \sum_{n_1, n_2, m_1, m_2} \frac{a_{n_1} \bar{a}_{n_2}}{(m_1 m_2 n_1 n_2)^{\frac{1}{2}}} \int_{\mathbb{R}} \left(\frac{m_1 n_2}{m_2 n_1} \right)^{it} W\left(\frac{2\pi m_1 m_2}{t} \right) \phi\left(\frac{t}{T} \right) dt + O(T^{\frac{1}{3}+\varepsilon}) \\ &= \mathcal{D} + \mathcal{S} + O(T^{\frac{1}{3}+\varepsilon}), \end{aligned}$$

where the sum is over $n_1, n_2 \leq N$, \mathcal{D} is the sum when $m_1 n_2 = m_2 n_1$, and \mathcal{S} is the sum when $m_1 n_2 \neq m_2 n_1$.

3.1. Diagonal terms. In this subsection, we consider the diagonal terms $m_1 n_2 = m_2 n_1$. For $j = 1, 2$, we write

$$m_j = \ell n_j^*, \quad \text{where } n_j^* = \frac{n_j}{(n_1, n_2)}.$$

The contribution of the diagonal term is

$$\begin{aligned}
 (3.1) \quad \mathcal{D} &= 2 \sum_{n_1, n_2, \ell} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{\ell n_1 n_2} \int_{\mathbb{R}} W\left(\frac{2\pi \ell^2 n_1^* n_2^*}{t}\right) \phi\left(\frac{t}{T}\right) dt \\
 &= \frac{2}{2\pi i} \sum_{n_1, n_2, \ell} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{\ell n_1 n_2} \int_{\mathbb{R}} \int_{(2)} \left(\frac{2\pi \ell^2 n_1^* n_2^*}{t}\right)^{-w} G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt \\
 &= \frac{2}{2\pi i} \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \\
 &\quad \times \int_{\mathbb{R}} \int_{(2)} \left(\frac{t}{2\pi n_1^* n_2^*}\right)^w \zeta(1+2w) G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt.
 \end{aligned}$$

This term will be later combined with a contribution from the off-diagonal terms. Together, they give the main term in Theorem 1.

3.2. Off-diagonal terms. In this subsection, we consider the terms with $m_1 n_2 \neq m_2 n_1$. We write

$$m_1 n_2 - m_2 n_1 = \Delta.$$

As $W(x) \ll x^{-A}$ when $x \gg 1$, we can truncate the sum over m_1, m_2 to when $m_1 m_2 \leq T^{1+\varepsilon}$. We introduce a smooth partition of unity

$$(3.2) \quad 1 = \sum_M' F_M(x), \quad T^{-100} \leq x \leq T^{1+\varepsilon},$$

where $F_M(x)$ is smooth, supported in $[\frac{M}{2}, 3M]$, and it satisfies

$$F_M^{(j)}(x) \ll_j \frac{1}{M^j} \quad \text{for all } j \geq 0.$$

Moreover, we can choose a partition of unity which satisfies

$$\sum_M' 1 \ll \log(2+T).$$

Therefore

$$\begin{aligned}
 (3.3) \quad \mathcal{S} &= 2 \sum_{N_1}' \sum_{N_2}' \sum_M' \sum_{\Delta \neq 0} \sum_{\substack{n_1, n_2, m_1, m_2 \\ m_1 n_2 - m_2 n_1 = \Delta}} \frac{a_{n_1} \bar{a}_{n_2}}{(m_1 m_2 n_1 n_2)^{\frac{1}{2}}} \\
 &\quad \times \left(\int_{\mathbb{R}} \left(1 + \frac{\Delta}{m_2 n_1}\right)^{it} W\left(\frac{2\pi m_1 m_2}{t}\right) \phi\left(\frac{t}{T}\right) dt \right) \\
 &\quad \times F_{N_1}(n_1) F_{N_2}(n_2) F_M(m_2) + O(1),
 \end{aligned}$$

where $N_1, N_2 \leq N$ and $M \leq T^{1+\varepsilon}$.

Next we show that the terms with $|\Delta| > D$, $D := \frac{MN_1}{T^{1-\varepsilon}}$, give a negligible contribution. In fact,

$$\frac{d^\ell}{dt^\ell} W\left(\frac{2\pi m_1 m_2}{t}\right) \ll_{\ell, A} \frac{1}{t^\ell} \min\left(1, \left(\frac{2\pi m_1 m_2}{t}\right)^{-A}\right),$$

whence, integrating by part ℓ times, we have

$$\begin{aligned}
& \sum'_{N_1, N_2, M} \sum_{|\Delta| > D} \sum_{n_1, m_2} \sum_{\substack{n_2, m_1 \\ m_1 n_2 - m_2 n_1 = \Delta}} \frac{a_{n_1} \bar{a}_{n_2}}{(m_1 m_2 n_1 n_2)^{\frac{1}{2}}} \\
& \quad \times \left(\int_{\mathbb{R}} \left(1 + \frac{\Delta}{m_2 n_1}\right)^{it} W\left(\frac{2\pi m_1 m_2}{t}\right) \phi\left(\frac{t}{T}\right) dt \right) F_{N_1}(n_1) F_{N_2}(n_2) F_M(m_2) \\
& \ll_{\ell} \sum'_{N_1, M} \sum_{|\Delta| > D} \sum_{\substack{n_1 \sim N_1 \\ m_2 \sim M}} \sum_{\substack{m_1 \leq T^{1+\varepsilon}, n_2 \leq N \\ m_1 n_2 - m_2 n_1 = \Delta}} \frac{T^{-\ell+1+\varepsilon}}{(m_1 m_2 n_1 n_2)^{\frac{1}{2}}} \left| \log\left(1 + \frac{\Delta}{m_2 n_1}\right) \right|^{-\ell} \\
& \ll_{\ell} \sum'_{N_1, M} \sum_{|\Delta| > D} \sum_{\substack{n_1 \sim N_1 \\ m_2 \sim M}} \frac{1}{\sqrt{n_1 m_2}} \sum_{\substack{m_1 \leq T^{1+\varepsilon}, n_2 \leq N \\ m_1 n_2 - m_2 n_1 = \Delta}} \frac{1}{\sqrt{n_2 m_1}} T^{-\ell+1+\varepsilon} \cdot \left(\frac{m_2 n_1}{|\Delta|}\right)^{\ell} \\
& \ll_{A, \varepsilon} T^{-A},
\end{aligned}$$

where ℓ is large enough.

Now, if $|\Delta| < D$, then $\frac{\Delta}{m_2 n_1} \ll \frac{1}{T^{1-\varepsilon}}$, and

$$m_1 = \frac{m_2 n_1 + \Delta}{n_2} = m_2 \frac{n_1}{n_2} \left(1 + \frac{\Delta}{m_2 n_1}\right).$$

Hence for $T < t < 2T$,

$$\begin{aligned}
\frac{1}{m_1} &= \frac{n_2}{m_2 n_1} \left(1 - \frac{\Delta}{m_2 n_1} + O\left(\frac{1}{T^{2-\varepsilon}}\right)\right), \\
\left(1 + \frac{\Delta}{m_2 n_1}\right)^{it} &= e^{it \log(1 + \frac{\Delta}{m_2 n_1})} = e^{it \frac{\Delta}{m_2 n_1}} \left(1 - \frac{it \Delta^2}{2m_2^2 n_1^2} + O\left(\frac{1}{T^{2-\varepsilon}}\right)\right), \\
W\left(\frac{2\pi m_1 m_2}{t}\right) &= W\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) + \frac{2\pi m_2 \Delta}{t n_2} W'\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) + O\left(\frac{1}{T^{2-\varepsilon}}\right).
\end{aligned}$$

Since $m_1 m_2 \leq T^{1+\varepsilon}$, we have $m_2(m_2 n_1 + \Delta) \leq n_2 T^{1+\varepsilon}$. Hence

$$M \ll T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}},$$

and the error term from using the above approximations in (3.3) is

$$\begin{aligned}
& \ll \frac{T}{T^{2-\varepsilon}} \sum'_{N_1} \sum'_{N_2} \sum'_{M \ll T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq D} \sum_{\substack{n_1 \sim N_1 \\ n_2 \sim N_2}} \sum_{m_2 \sim M} \frac{1}{m_2 n_1} \\
& \ll \sum'_{N_1} \sum'_{N_2} \sum'_{M \ll T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}}} \frac{M N_1 N_2}{T^{2-\varepsilon}} \ll \frac{\sqrt{T} N_2^{\frac{3}{2}} N_1^{\frac{1}{2}}}{T^{2-\varepsilon}} \ll \frac{N^2}{T^{\frac{3}{2}-\varepsilon}},
\end{aligned}$$

using that $D = M N_1 / T^{1-\varepsilon}$ and that $M \ll T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}}$. Thus, we have

$$\mathfrak{S} = \mathfrak{A} + \mathfrak{E} + O\left(1 + \frac{N^2}{T^{\frac{3}{2}-\varepsilon}}\right),$$

where

$$(3.4) \quad \mathcal{A} = 2 \sum'_{N_1} \sum'_{N_2} \sum'_{M \leq T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq D} \sum_{n_1, n_2} \sum_{\substack{n_1 m_2 \equiv -\Delta \pmod{n_2} \\ m_2 > 0}} \frac{a_{n_1} \bar{a}_{n_2}}{m_2 n_1} \\ \times \left(\int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi m_2 n_1}\right) W\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) \phi\left(\frac{t}{T}\right) dt \right) \\ \times F_M(m_2) F_{N_1}(n_1) F_{N_2}(n_2),$$

and

$$(3.5) \quad \mathcal{E} = 2 \sum'_{N_1} \sum'_{N_2} \sum'_{M \leq T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq D} \sum_{n_1, n_2} \sum_{\substack{n_1 m_2 \equiv -\Delta \pmod{n_2} \\ m_2 > 0}} \frac{a_{n_1} \bar{a}_{n_2}}{m_2 n_1} \\ \times F_M(m_2) F_{N_1}(n_1) F_{N_2}(n_2) \\ \times \int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi m_2 n_1}\right) \left[W\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) \left(-\frac{\Delta}{2m_2 n_1} - \frac{it\Delta^2}{2m_2^2 n_1^2} \right) \right. \\ \left. + \frac{2\pi m_2 \Delta}{t n_2} W'\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) \right] \phi\left(\frac{t}{T}\right) dt,$$

since the rest of the terms arising from the above approximations also give a contribution which is $O(N^2 T^{-\frac{3}{2} + \varepsilon})$.

First, we consider \mathcal{A} . Giving an eligible bound for \mathcal{E} is easy and we will do it in the next section. Extracting the common divisor d of n_1 and n_2 , we re-write the sum (3.4) as

$$\mathcal{A} = 2 \sum_{d \leq N} \frac{1}{d} \sum'_{N_1, N_2 \leq N} \sum'_{M \leq T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} a_{dn_1} \bar{a}_{dn_2} \\ \times F_{N_1}(dn_1) F_{N_2}(dn_2) \mathcal{A}_{M, N_i}(n_1, n_2, \Delta),$$

where

$$\mathcal{A}_{M, N_i}(n_1, n_2, \Delta) = \sum_{m_2 \equiv -\bar{n}_1 \Delta \pmod{n_2}} \frac{F_M(m_2)}{m_2 n_1} \\ \times \left(\int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi m_2 n_1}\right) W\left(\frac{2\pi m_2^2 n_1}{t n_2}\right) \phi\left(\frac{t}{T}\right) dt \right).$$

By Poisson summation formula,

$$\mathcal{A}_{M, N_i}(n_1, n_2, \Delta) = \frac{1}{n_1 n_2} \sum_{h \in \mathbb{Z}} e\left(-\frac{h \bar{n}_1 \Delta}{n_2}\right) \int_0^\infty e\left(-\frac{hx}{n_2}\right) \frac{F_M(x)}{x} \\ \times \int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi x n_1}\right) W\left(\frac{2\pi x^2 n_1}{t n_2}\right) \phi\left(\frac{t}{T}\right) dt dx.$$

After the change of variable $x \rightarrow \frac{x}{n_1}$, this becomes

$$\mathcal{A}_{M, N_i}(n_1, n_2, \Delta) = \frac{1}{n_1 n_2} \sum_{h \in \mathbb{Z}} \tilde{\mathcal{A}}_{M, N_i}(h, n_1, n_2, \Delta) e\left(-\frac{h \bar{n}_1 \Delta}{n_2}\right),$$

where

$$\widetilde{\mathcal{A}}_{M,N_i}(h, n_1, n_2, \Delta) = \int_0^\infty e\left(-\frac{hx}{n_1 n_2}\right) \frac{F_M\left(\frac{x}{n_1}\right)}{x} \int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi x}\right) W\left(\frac{2\pi x^2}{n_1 n_2 t}\right) \phi\left(\frac{t}{T}\right) dt dx.$$

To understand the contribution of $\widetilde{\mathcal{A}}_{M,N_i}(h, n_1, n_2, \Delta)$, we consider the following three cases.

Case 1: $h = 0$. The contribution to \mathcal{A} from $h = 0$ is

$$(3.6) \quad \begin{aligned} \mathcal{A}_0 = 2 \sum_{d \leq N} \sum'_{N_1 \leq N} \sum'_{N_2 \leq N} \sum'_{M \leq T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \\ \times \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2} F_{N_1}(dn_1) F_{N_2}(dn_2)}{dn_1 n_2} \\ \times \int_{\mathbb{R}} \int_0^\infty F_M\left(\frac{x}{n_1}\right) e\left(\frac{\Delta t}{2\pi x}\right) W\left(\frac{2\pi x^2}{n_1 n_2 t}\right) \frac{dx}{x} \phi\left(\frac{t}{T}\right) dt. \end{aligned}$$

Now, we can extend the sum over Δ to $\Delta \in \mathbb{Z} \setminus \{0\}$, since it can be shown as before that the terms $|\Delta| \geq \frac{D}{d}$ give a negligible contribution. Making the change of variables $y = \frac{t}{x}$ and integrating by parts twice we see that the second line of (3.6) is equal to

$$\begin{aligned} & -\frac{1}{\Delta^2} \int_{\mathbb{R}} \int_0^\infty e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(F_M\left(\frac{t}{n_1 y}\right) W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt \\ & = -\frac{1}{\Delta^2} \int_{\mathbb{R}} \int_{R(t, n_1)} e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(F_M\left(\frac{t}{n_1 y}\right) W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt + O\left(\frac{1}{\Delta^2}\right), \end{aligned}$$

where

$$R(t, n_1) = \left\{ y : T^{-100} < \frac{t}{n_1 y} < T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}} \right\}$$

and where we estimated trivially the part of the integral over y with $y \in \mathbb{R}_{>0} \setminus R(t, n_1)$, using the properties of W and F_M (and $n_1 \ll T$). Thus, summing over M we have

$$\begin{aligned} & \sum'_M \int_{\mathbb{R}} \int_0^\infty F_M\left(\frac{x}{n_1}\right) e\left(\frac{\Delta t}{2\pi x}\right) W\left(\frac{2\pi x^2}{n_1 n_2 t}\right) \frac{dx}{x} \phi\left(\frac{t}{T}\right) dt \\ & = -\frac{1}{\Delta^2} \int_{\mathbb{R}} \int_{R(t, n_1)} e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt + O\left(\frac{\log(2+T)}{\Delta^2}\right) \\ & = -\frac{1}{\Delta^2} \int_{\mathbb{R}} \int_0^\infty e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt + O\left(\frac{\log(2+T)}{\Delta^2}\right). \end{aligned}$$

Therefore, summing over N_1, N_2 , we have

$$\begin{aligned} \mathcal{A}_0 = -2 \sum_{d \leq N} \sum_{|\Delta| \neq 0} \sum_{\substack{n_1, n_2 \leq \frac{N}{d} \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2}}{dn_1 n_2 \Delta^2} \\ \times \int_{\mathbb{R}} \int_0^\infty e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt + O(T^\varepsilon) \end{aligned}$$

$$\begin{aligned}
&= -2 \sum_{d \leq N} \sum_{\substack{n_1, n_2 \leq \frac{N}{d} \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2}}{dn_1 n_2} \\
&\quad \times \int_{\mathbb{R}} \int_0^\infty \sum_{|\Delta| \neq 0} \frac{1}{\Delta^2} e\left(\frac{\Delta y}{2\pi}\right) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt + O(T^\varepsilon) \\
&= \mathcal{A}_{0,+} + \mathcal{A}_{0,-} + O(T^\varepsilon),
\end{aligned}$$

where we can take the sum over Δ inside the integrals since they converge absolutely and, after a change of variables,

$$\begin{aligned}
\mathcal{A}_{0,\pm} &= -2 \sum_{d \leq N} \sum_{\substack{n_1, n_2 \leq \frac{N}{d} \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2}}{dn_1 n_2} \\
&\quad \times \int_{\mathbb{R}} \int_0^\infty \sum_{\Delta=1}^\infty (e^{iy} + e^{-iy}) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi \Delta^2 t}{n_1 n_2 y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt \\
&= -2 \sum_{n_1, n_2 \leq N} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \\
&\quad \times \int_{\mathbb{R}} \int_0^\infty \sum_{\Delta=1}^\infty 2 \cos(y) \frac{d^2}{dy^2} \left(W\left(\frac{2\pi \Delta^2 t}{n_1^* n_2^* y^2}\right) \frac{1}{y} \right) dy \phi\left(\frac{t}{T}\right) dt,
\end{aligned}$$

where we recall that

$$n_i^* = \frac{n_i}{(n_1, n_2)} \quad \text{for } i = 1, 2.$$

We notice that

$$\begin{aligned}
&\sum_{\Delta=1}^\infty \frac{d^2}{dy^2} \left(W\left(\frac{2\pi \Delta^2 t}{n_1^* n_2^* y^2}\right) \frac{1}{y} \right) \\
&= \frac{1}{2\pi i} \sum_{\Delta=1}^\infty \int_{(2)} \left(\frac{2\pi \Delta^2 t}{n_1^* n_2^*} \right)^{-w} (2w-1)(2w-2) y^{2w-3} G(w) \frac{dw}{w} \\
&= \frac{1}{2\pi i} \int_{(\frac{5}{4})} \zeta(2w) \left(\frac{2\pi t}{n_1^* n_2^*} \right)^{-w} (2w-1)(2w-2) y^{2w-3} G(w) \frac{dw}{w}.
\end{aligned}$$

For $0 < \Re(s) < 1$, we have

$$\int_0^\infty \cos(y) y^{s-1} dy = \Gamma(s) \cos\left(\frac{\pi s}{2}\right)$$

(see, for example, [11, formula 3.381, 5, p. 346]), whence we are left with

$$\frac{-2}{2\pi i} \int_{(\frac{1}{4})} \Gamma(2w) \cos(\pi w) \zeta(2w) \left(\frac{2\pi t}{n_1^* n_2^*} \right)^{-w} G(w) \frac{dw}{w},$$

where we used the multiplication formula for the gamma function, the identity

$$\cos(x - \pi) = -\cos(x),$$

and we moved the line of integration without encountering any pole, due to the assumption

that $G(w)$ vanishes at $w = \frac{1}{2}$. Thus,

$$\begin{aligned} \mathcal{A}_0 + O(T^\varepsilon) &= \frac{2}{2\pi i} \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \\ &\quad \times \int_{\mathbb{R}} \int_{(\frac{1}{4})} 2 \cos(\pi w) \Gamma(2w) \zeta(2w) \left(\frac{2\pi t}{n_1^* n_2^*} \right)^{-w} G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt \\ &= \frac{-2}{2\pi i} \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \\ &\quad \times \int_{\mathbb{R}} \int_{(-\frac{1}{4})} \zeta(1+2w) \left(\frac{t}{2\pi n_1^* n_2^*} \right)^w G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt, \end{aligned}$$

where we used the functional equation of the Riemann zeta-function (e.g. [6, Chapter 10]), and then we made the change of variables $w \rightarrow -w$ and use the fact that $G(w) = G(-w)$.

From (3.1), we have

$$\begin{aligned} \mathcal{D} + \mathcal{A}_0 &= \frac{2}{2\pi i} \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \int_{\mathbb{R}} \int_{(2)} \left(\frac{t}{2\pi n_1^* n_2^*} \right)^w \zeta(1+2w) G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt \\ &\quad - \frac{2}{2\pi i} \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \\ &\quad \times \int_{\mathbb{R}} \int_{(-\frac{1}{4})} \left(\frac{t}{2\pi n_1^* n_2^*} \right)^w \zeta(1+2w) G(w) \frac{dw}{w} \phi\left(\frac{t}{T}\right) dt + O(T^\varepsilon) \\ &= \sum_{n_1, n_2} \frac{a_{n_1} \bar{a}_{n_2}(n_1, n_2)}{n_1 n_2} \int_{\mathbb{R}} \left(\log \frac{t}{2\pi n_1^* n_2^*} + 2\gamma \right) \phi\left(\frac{t}{T}\right) dt + O(T^\varepsilon), \end{aligned}$$

since

$$\text{Res}_{w=0} x^w \zeta(1+2w) \frac{G(w)}{w} = \frac{1}{2} \log x + \gamma.$$

Now we have the main term. The rest of the off-diagonal terms contribute to the error term as shown in the following two cases.

Case 2: $|h| \geq \frac{N_2}{dM} T^\varepsilon$. In this case and in Case 3, we define

$$H_d := \frac{N_2}{dM} T^\varepsilon.$$

By changing variable $t = xy$, we have

$$\begin{aligned} &\frac{1}{n_1 n_2} \widetilde{\mathcal{A}}_{M, N_i}(h, n_1, n_2, \Delta) \\ &= \frac{1}{n_1 n_2} \int_{\mathbb{R}} e\left(\frac{\Delta y}{2\pi}\right) \int_0^\infty e\left(-\frac{hx}{n_1 n_2}\right) F_M\left(\frac{x}{n_1}\right) W\left(\frac{2\pi x}{n_1 n_2 y}\right) \phi\left(\frac{yx}{T}\right) dx dy. \end{aligned}$$

Since F_M is supported in $[\frac{M}{2}, 3M]$, one has

$$x \asymp \frac{N_1 M}{d}.$$

Moreover,

$$\frac{y}{T} \asymp \frac{1}{x} \asymp \frac{d}{N_1 M}$$

due to the support of ϕ , and

$$\frac{1}{n_1 n_2 y} \ll \frac{T^{\varepsilon_1}}{x} \asymp \frac{dT^{\varepsilon_1}}{N_1 M}$$

because of the rapid decay of W . Hence integrating by parts $\ell + 1$ times, for $T \leq t \leq 2T$ we have

$$\begin{aligned} & \frac{1}{n_1 n_2} \int_0^\infty e\left(-\frac{hx}{n_1 n_2}\right) F_M\left(\frac{x}{n_1}\right) W\left(\frac{2\pi x}{n_1 n_2 y}\right) \phi\left(\frac{yx}{T}\right) dx \\ & \ll_{\ell, \varepsilon} \frac{d^2}{N_1 N_2} \left(\frac{n_1 n_2}{h} \frac{dT^{\varepsilon_1}}{MN_1}\right)^{\ell+1} \frac{MN_1}{d} \\ & \ll \left(\frac{T^{\varepsilon_1}}{h}\right)^{\ell+1} \left(\frac{N_2}{dM}\right)^\ell. \end{aligned}$$

Therefore, the contribution to \mathfrak{S} when $|h| > H_d$ is

$$\begin{aligned} & \ll \sum_{d \leq N} \sum'_{\substack{N_1, N_2 \leq N \\ M \leq T^{\frac{1}{2} + \varepsilon} \sqrt{\frac{N_2}{N_1}}}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2} F_{N_1}(dn_1) F_{N_2}(dn_2)}{d} \\ & \quad \times \sum_{|h| \geq H_d} \frac{dT}{h N_1 M} \left(\frac{N_2 T^{\varepsilon_1}}{d M h}\right)^\ell \\ & \ll T^{-A}, \end{aligned}$$

when ℓ is sufficiently large. Thus, the terms $|h| > H_d$ give a negligible contribution.

Case 3: $0 < |h| < H_d$. It is sufficient to consider the terms $0 < h < H_d$. By changing variables $t = yx$, and x to $x n_1 n_2$, we will consider the dyadic contribution

$$\begin{aligned} \mathcal{A}_{M, N_1, N_2}^* &:= \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{0 < h < H_d} \frac{a_{dn_1} \bar{a}_{dn_2} F_{N_1}(dn_1) F_{N_2}(dn_2)}{d} e\left(-\frac{h \bar{n}_1}{n_2}\right) \\ & \quad \times \int_0^\infty e(-hx) F_M(x n_2) \int_{\mathbb{R}} e\left(\frac{\Delta y}{2\pi}\right) W\left(\frac{2\pi x}{y}\right) \phi\left(\frac{yx n_1 n_2}{T}\right) dy dx. \end{aligned}$$

We write ϕ in term of its Mellin transform $\tilde{\phi}$, to separate the variables n_1 and n_2 . Let

$$h\Delta = a, \quad A = \frac{DH_d}{d} = \frac{N_1 N_2}{d^2 T^{1-\varepsilon}}, \quad v_{x,y}(a) = \sum_{h\Delta=a} e\left(-hx + \frac{\Delta y}{2\pi}\right).$$

Therefore we have

$$\begin{aligned} \mathcal{A}_{M, N_1, N_2}^* &= \frac{1}{(2\pi i)d} \int_0^\infty \int_{\mathbb{R}} \int_{(\varepsilon)} W\left(\frac{2\pi x}{y}\right) \sum_{0 < |a| < A} v_{x,y}(a) \\ & \quad \times \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} \frac{a_{dn_1} \bar{a}_{dn_2} F_{N_1}(dn_1) F_{N_2}(dn_2) F_M(x n_2)}{n_1^w n_2^w} \\ & \quad \times e\left(-\frac{a \bar{n}_1}{n_2}\right) \tilde{\phi}(w) \frac{T^w}{x^w y^w} dw dy dx. \end{aligned}$$

Since F_M is supported in $[\frac{M}{2}, 3M]$, one has

$$x \asymp \frac{dM}{N_2}.$$

Moreover,

$$y \asymp \frac{T}{xn_1n_2} \asymp \frac{Td}{MN_1}$$

because ϕ is supported in the interval $[1, 2]$. Thus, using Proposition 1, we have

$$(3.7) \quad \mathcal{A}_{M,N_1,N_2}^* \ll \frac{1}{d} \int_{x \asymp \frac{dM}{N_2}} \int_{y \asymp \frac{Td}{MN_1}} \left(\frac{(N_1N_2A)^{\frac{17}{20}+\varepsilon}}{d^{\frac{17}{10}-\varepsilon}} \frac{(N_1+N_2)^{\frac{1}{4}}}{d^{\frac{1}{4}}} \right. \\ \left. + \frac{(N_1N_2A)^{\frac{7}{8}+\varepsilon}}{d^{\frac{7}{4}-\varepsilon}} \frac{(AN_1+AN_2)^{\frac{1}{8}}}{d^{\frac{1}{8}}} \right) dy dx \\ \ll \left(T^{\frac{3}{20}+\varepsilon} \frac{(N_1N_2)^{\frac{7}{10}}}{d^{\frac{53}{20}}} (N_1+N_2)^{\frac{1}{4}} + T^\varepsilon \frac{(N_1N_2)^{\frac{7}{8}}}{d^{\frac{23}{8}}} (N_1+N_2)^{\frac{1}{8}} \right)$$

Summing over dyadic intervals for

$$M \leq T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}}, \quad N_i \leq N \quad \text{and} \quad d \leq N,$$

we have that the contribution to \mathcal{A} when $|h| \leq H_d$ is bounded by $T^{\frac{3}{20}+\varepsilon} N^{\frac{33}{20}} + T^\varepsilon N^{\frac{15}{8}}$. Therefore we take N up to $T^{\frac{17}{33}-\varepsilon}$ to obtain an eligible error term in Theorem 1.

3.3. A trivial bound for \mathcal{E} . Extracting the common divisor d from n_1 and n_2 , applying Poisson summation formula, and changing variables, we can write (3.5) as

$$\mathcal{E} = 2 \sum_{d \leq N} \frac{1}{d} \sum'_{N_1, N_2 \leq N} \sum'_{M \leq T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2}{N_1}}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{\substack{n_1, n_2 \\ (n_1, n_2) = 1}} a_{dn_1} \bar{a}_{dn_2} \\ \times F_{N_1}(dn_1) F_{N_2}(dn_2) \mathcal{E}_{M, N_i}(n_1, n_2, \Delta),$$

where

$$\mathcal{E}_{M, N_i}(n_1, n_2, \Delta) = \frac{1}{n_1 n_2} \sum_{h \in \mathbb{Z}} e\left(-\frac{h \bar{n}_1 \Delta}{n_2}\right) \int_0^\infty e\left(-\frac{hx}{n_1 n_2}\right) F_M\left(\frac{x}{n_1}\right) \int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi x}\right) \phi\left(\frac{t}{T}\right) \\ \times \left[W\left(\frac{2\pi x^2}{n_1 n_2 t}\right) \left(-\frac{\Delta}{2x^2} - \frac{it\Delta^2}{2x^3}\right) + \frac{2\pi\Delta}{n_1 n_2 t} W'\left(\frac{2\pi x^2}{n_1 n_2 t}\right) \right] dt dx.$$

Integrating by parts, as in Case 2 of the previous section, we see that the contribution coming from the terms with $|h| > H_d$ is $O(1)$. Thus, estimating trivially the rest of the terms we have

$$\mathcal{E}_{M, N_i}(n_1, n_2, \Delta) \ll \frac{T^\varepsilon}{n_1 n_2} \left(1 + \frac{N_2}{dM}\right),$$

whence

$$\mathcal{E} \ll T^{-\frac{1}{2}+\varepsilon} N + T^{-1+\varepsilon} N^2 \ll NT^\varepsilon$$

and the proof of Theorem 1 is complete.

3.4. The proof of Theorem 2. The proof of Theorem 2 is the same as Theorem 1 except that we use (1.3) instead of Proposition 1 in (3.7). Notice that (1.3) is applicable, since

$$A = \frac{N_1 N_2}{d^2 T^{1-\varepsilon}} \leq \left(\frac{N_1 N_2}{d^2} \right)^{\frac{\frac{1}{2}-r}{1+2t} + \varepsilon}$$

by

$$\frac{N_i}{d} \leq N \leq T^{\frac{1}{2} + \frac{\frac{1}{2}-r}{1+2(r+2t)}}.$$

Thus, we obtain that

$$\begin{aligned} \mathcal{A}_{M,N_1,N_2}^* &\ll \frac{T^\varepsilon}{d} \int_{x \asymp \frac{dM}{N_2}} \int_{y \asymp \frac{Td}{MN_1}} \left(\frac{\sqrt{N_1 N_2 A}}{d} \left(\frac{(N_1 + N_2)^{\frac{1}{2}+r} A^t}{d^{\frac{1}{2}+r}} \right) + \frac{AN_1}{d} + \frac{AN_2}{d} \right) dy dx \\ &\ll T^{\frac{1}{2}+\varepsilon-t} \left(\frac{(N_1 + N_2)^{\frac{1}{2}+r} (N_1 N_2)^t}{d^{\frac{3}{2}+r+2t}} \right) + \frac{N_1}{d^2} + \frac{N_2}{d^2}. \end{aligned}$$

Summing over dyadic intervals for M, N_i , we have that the contribution to \mathcal{A} from these terms is $T^{\frac{1}{2}+\varepsilon-t} N^{\frac{1}{2}+r+2t} + N^{1+\varepsilon}$ and Theorem 2 follows.

4. Proof of Theorem 3

The proof of Theorem 3 follows the proof of Theorem 1 except the last part when $0 < |h| < H_d$. Here we only modify the last part of the proof using the same arguments by Deshouillers and Iwaniec in [8]. By the same change of variables, we have to consider

$$\begin{aligned} \mathcal{A}_{M,N_1,N_2}^* &= \sum_{\substack{n_1, n_2 \\ (n_1, n_2)=1}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{0 < |h| < H_d} a_{dn_1} \bar{a}_{dn_2} F_{N_1}(dn_1) F_{N_2}(dn_2) e\left(-\frac{h\bar{n}_1 \Delta}{n_2}\right) \\ &\quad \times \int_0^\infty e(-hx) F_M(xn_2) \int_{\mathbb{R}} e\left(\frac{\Delta y}{2\pi}\right) W\left(\frac{2\pi x}{y}\right) \phi\left(\frac{xy n_1 n_2}{T}\right) dy dx. \end{aligned}$$

We now write a_{dn_1} as $\alpha_{\mu j} \beta_{vr}$, where $\mu \mid d^\infty$, $(d, j) = 1$, $n_1 = \varrho r j$, $v = \frac{d}{(\mu, d)}$ and $\varrho = \frac{\mu}{(\mu, d)}$. Therefore, we have to bound

$$\begin{aligned} (4.1) \quad &\sum_{d \leq T} \frac{1}{d} \sum'_{N_1, N_2, M} \sum_{\substack{\mu \mid d^\infty \\ v = \frac{d}{(\mu, d)}}} \sum_{0 < |\Delta| \leq \frac{D}{d}} \sum_{0 < |h| < H_d} \sum_{(n_2, \varrho)=1} \bar{a}_{dn_2} \\ &\quad \times \sum_{(j, dn_2)=1} \alpha_{\mu j} \sum_{(r, n_2)=1} \beta_{vr} e\left(-\frac{h\Delta \overline{\varrho r j}}{n_2}\right) F_{N_1}(d\varrho r j) F_{N_2}(dn_2) \\ &\quad \times \int_0^\infty e(-hx) F_M(xn_2) \int_{\mathbb{R}} e\left(\frac{\Delta y}{2\pi}\right) W\left(\frac{2\pi x}{y}\right) \phi\left(\frac{xy \varrho r j n_2}{T}\right) dy dx, \end{aligned}$$

where the sums over N_1, N_2, M are dyadic sums up to NK, NK , and $T^{\frac{1}{2}+\varepsilon} \sqrt{N_2/N_1}$.

To bound the above sum, we use Proposition 2. However, first we need to apply Mellin's transform to F_{N_1} and ϕ to separate variables n_2, r, j . The technique is standard, so we skip the details. From Proposition 2, the sum over ℓ is the sum over n_2 , and $L = N_2/d$. The sum over j is the sum over j , and $J \leq N/\mu$. The sum over u is the sum over $h\Delta$,

and $U = N_1 N_2 / (d^2 T^{1-2\epsilon})$. Finally, the sum over v is the sum over r , and $V \leq K/v$. Moreover, we note that $JV \leq N_1/(d\varrho)$. Applying Proposition 2, we obtain that (4.1) is bounded by (after summing over dyadic M)

$$\begin{aligned}
&\ll T^\epsilon \sum_{d \leq T} \sum_{\substack{\mu | d^\infty \\ v = \frac{d}{(\mu, d)}}} \sum'_{N_1} \sum'_{N_2} \frac{dT}{N_1 N_2} \left(\frac{N_1 N_2}{\varrho^{\frac{1}{2}} d^2 T^{\frac{1}{2}}} \right) \left\{ \left(\frac{N_2 N}{d\mu} \right)^{\frac{1}{2}} + \left(\frac{N_1 N_2}{d^2 T} + \frac{K}{v} \right)^{\frac{1}{4}} \right. \\
&\quad \times \left[\frac{N_2 N}{d\mu} \left(\frac{N_1 N_2}{T d^2} + \frac{\varrho K}{v} \right) \left(\frac{N_2}{d} + \frac{\varrho K^2}{v^2} \right) + \frac{\varrho N_1 N_2 N^2 K^2}{d^2 T \mu^2 v^2} \right]^{\frac{1}{4}} \Big\} \\
&\ll \sum_{d \leq T} \sum_{\substack{\mu | d^\infty \\ v = \frac{d}{(\mu, d)}}} \frac{T^{\frac{1}{2} + \epsilon}}{\varrho^{\frac{1}{2}} d} \left\{ \left(\frac{N^2 K}{d\mu} \right)^{\frac{1}{2}} + \left(\frac{N^2 K^2}{d^2 T} + \frac{K}{v} \right)^{\frac{1}{4}} \right. \\
&\quad \times \left[\frac{N^2 K}{d\mu} \left(\frac{N^2 K^2}{T d^2} + \frac{\varrho K}{v} \right) \left(\frac{NK}{d} + \frac{\varrho K^2}{v^2} \right) + \frac{\varrho N^4 K^4}{d^2 T \mu^2 v^2} \right]^{\frac{1}{4}} \Big\} \\
&\ll T^\epsilon \left(T^{\frac{1}{2}} N^{\frac{3}{4}} K + T^{\frac{1}{2}} N K^{\frac{1}{2}} + N^{\frac{7}{4}} K^{\frac{3}{2}} \right) \sum_{d \leq T} \sum_{\substack{\mu | d^\infty \\ v = \frac{d}{(\mu, d)}}} \frac{1}{d^{\frac{5}{4}}} \frac{1}{\mu^{\frac{1}{4}}} \\
&\ll T^\epsilon \left(T^{\frac{1}{2}} N^{\frac{3}{4}} K + T^{\frac{1}{2}} N K^{\frac{1}{2}} + N^{\frac{7}{4}} K^{\frac{3}{2}} \right),
\end{aligned}$$

and this completes the proof of Theorem 3.

5. Proof of Theorem 4

The proof of Theorem 4 follows the proof of Theorem 1 except the last part when $0 < |h| < H_d$. We recall that we have

$$a_b = \sum_{nk=b} \alpha_n \beta_k,$$

and we assume that α_n is supported on $[NT^{-\xi_1}, 2N]$, where $N \ll T^{\frac{1}{2} + \epsilon}$ and $0 \leq \xi_1 \leq \frac{1}{5}$. Moreover, β_k is supported on $k \in [KT^{-\xi_2}, 2K]$, and $0 \leq \xi_2 \leq \frac{1}{16}$. Let $\xi_1 + \xi_2 = \xi$. We introduce smooth partitions of unity in the sums over n and k (without indicating it, to save notation). Thus, $n_i \approx N_i$ and $k_i \approx K_i$ (note that in the notation of Section 3 N_i was the size of $b = nk$, so $N_i K_i$ in the current notation). Thus, $db_i \asymp N_i K_i$. Moreover, we assume that $\alpha_n = \psi(n)$, where $\psi(x)$ is a smooth function such that $\psi^{(j)}(x) \ll_j x^{-j}$ for $j \geq 0$.

We have to bound

$$\begin{aligned}
\mathcal{A}'_{N,K} &:= \sum_{d \leq NK} \sum'_{N_1, N_2, K_1, K_2} \sum'_{M \leq T^{\frac{1}{2} + \epsilon}} \sum_{\substack{b_1, b_2 \\ (b_1, b_2) = 1}} \sum_{0 < |\Delta| \leq \frac{MN_1 K_1}{dT^{1-\epsilon}}} \\
&\quad \times \sum_{0 < |h| < \frac{N_2 K_2 T^\epsilon}{dM}} \frac{a_d b_1 \overline{a_d b_2} F_{N_1 K_1}(db_1) F_{N_2 K_2}(db_2)}{db_1 b_2} e\left(-\frac{h \overline{b_1} \Delta}{b_2}\right) \\
&\quad \times \int_0^\infty e\left(-\frac{hx}{b_1 b_2}\right) \frac{F_M\left(\frac{x}{b_1}\right)}{x} \int_{\mathbb{R}} e\left(\frac{\Delta t}{2\pi x}\right) W\left(\frac{2\pi x^2}{b_1 b_2 t}\right) \phi\left(\frac{t}{T}\right) dt dx.
\end{aligned}$$

After the change of variables $y = \frac{t}{2\pi x}$ and then $z = \frac{x}{b_1 b_2}$, it becomes

$$\begin{aligned} \mathcal{A}'_{N,K} &= 2\pi \sum_{d \leq NK} \sum'_{N_1, N_2, K_1, K_2} \sum'_{M \leq T^{\frac{1}{2} + \varepsilon}} \sum_{\substack{b_1, b_2 \\ (b_1, b_2) = 1}} \sum_{0 < |\Delta| \leq \frac{MN_1 K_1}{dT^{1-\varepsilon}}} \\ &\times \sum_{0 < |h| < \frac{N_2 K_2 T^\varepsilon}{dM}} \frac{a_{db_1} \overline{a_{db_2}} F_{N_1 K_1}(db_1) F_{N_2 K_2}(db_2)}{d} e\left(-\frac{h \bar{b}_1 \Delta}{b_2}\right) \\ &\times \int_0^\infty e(-hz) F_M(zb_2) \int_{\mathbb{R}} e(\Delta y) W\left(\frac{z}{y}\right) \phi\left(\frac{2\pi y z b_1 b_2}{T}\right) dy dz. \end{aligned}$$

Firstly, we claim that we can truncate the sum over d at height

$$Y := \frac{(N_1 K_1 N_2 K_2)^{\frac{1}{2}}}{T^{\frac{1}{2} + \eta}}$$

for some small $\eta > 0$, up to an error term with a power saving. This is because for larger values of d we are essentially left with the contribution coming from a Dirichlet polynomial of length $T^{\frac{1}{2} + \eta}$, which we can bound using the method used to prove Theorem 1. More precisely, by (3.7), we have that the contribution from the terms with $d \geq Y$ is bounded by

$$\begin{aligned} (5.1) \quad &\ll \sum'_{N_1, N_2, K_1, K_2} \sum_{Y \leq d \leq NK} \left(\frac{T^{\frac{3}{20} + \varepsilon} (N_1 K_1 + N_2 K_2)^{\frac{1}{4}} (N_1 K_1 N_2 K_2)^{\frac{7}{10}}}{d^{\frac{53}{20}}} \right. \\ &\quad \left. + T^\varepsilon \frac{(N_1 K_1 + N_2 K_2)^{\frac{1}{8}} (N_1 K_1 N_2 K_2)^{\frac{7}{8}}}{d^{\frac{23}{8}}} \right) \\ &\ll \sum'_{N_1, N_2, K_1, K_2} \left(T^{\frac{39}{40} + \frac{33}{20}\eta + \varepsilon} \left(\frac{N_1 K_1}{N_2 K_2} + \frac{N_2 K_2}{N_1 K_1} \right)^{\frac{1}{8}} \right. \\ &\quad \left. + T^{\frac{15}{16} + \frac{15}{8}\eta + \varepsilon} \left(\frac{N_1 K_1}{N_2 K_2} + \frac{N_2 K_2}{N_1 K_1} \right)^{\frac{1}{16}} \right) \\ &\ll T^{\frac{39}{40} + \frac{33}{20}\eta + \frac{1}{8}\xi + \varepsilon} + T^{\frac{15}{16} + \frac{15}{8}\eta + \frac{1}{16}\xi + \varepsilon}, \end{aligned}$$

since

$$T^{-\xi} \ll \frac{N_1 K_1}{N_2 K_2} \ll T^\xi.$$

For the remaining part of the proof we use Watt's arguments in [19]. We write

$$\begin{aligned} a_{db_i} &= \sum_{h_i k_i = db_i} \alpha_{h_i} \beta_{k_i} \\ &= \sum_{\substack{f_i g_i = b_i, \mu_i v_i = d \\ (g_i, \frac{d}{v_i}) = 1}} \alpha_{f_i \mu_i} \beta_{g_i v_i}, \end{aligned}$$

so that

$$f_i \asymp \frac{N_i}{\mu_i} \quad \text{and} \quad g_i \asymp \frac{K_i}{v_i}.$$

We will apply Proposition 3 to bound

$$\begin{aligned}
 (5.2) \quad & \sum'_{M \leq T^{\frac{1}{2}+\varepsilon} \sqrt{\frac{N_2 K_2}{N_1 K_1}}} \sum_{d \leq Y} \frac{1}{d} \sum_{\substack{\mu_1, \nu_1, \mu_2, \nu_2 \\ \mu_1 \nu_1 = \mu_2 \nu_2 = d}} \int_{z \asymp \frac{dM}{N_2 K_2}} \int_{y \asymp \frac{dT}{MN_1 K_1}} \left| W\left(\frac{z}{y}\right) \right. \\
 & \times \sum_{\substack{g_1, g_2 \\ (g_i, \frac{d}{v_i})=1}} \sum_{\substack{f_1, f_2 \\ (f_1 g_1, f_2 g_2)=1}} \sum_{0 < |\Delta| \leq \frac{MN_1 K_1}{dT^{1-\varepsilon}}} \sum_{0 < |h| < \frac{N_2 K_2 T^\varepsilon}{dM}} \alpha_{f_1 \mu_1} \beta_{g_1 \nu_1} \\
 & \times \overline{\alpha_{f_2 \mu_2} \beta_{g_2 \nu_2}} e\left(\frac{h \Delta \overline{f_1 g_1}}{f_2 g_2}\right) F_{N_1 K_1}(f_1 g_1 \mu_1 \nu_1) F_{N_2 K_2}(f_2 g_2 \mu_2 \nu_2) \\
 & \times e(-hz) e(\Delta y) F_M(z f_2 g_2) \phi\left(\frac{2\pi y z f_1 f_2 g_1 g_2}{T}\right) \Big| dy dz.
 \end{aligned}$$

Before using Proposition 3 – with

$$H = \frac{MN_1 K_1}{dT^{1-\varepsilon}}, \quad C = \frac{N_2 K_2 T^\varepsilon}{dM}, \quad r = g_1, \quad s = g_2, \quad v = f_1, \quad p = f_2$$

in the proposition respectively – we verify that

$$X^2 = \frac{N_1 N_2 K_1 K_2}{H C d^2} = T^{1-\varepsilon} \gg T^\varepsilon,$$

which is clearly satisfied if ε is small enough, and that

$$\begin{aligned}
 \left(\frac{K_1 K_2}{\nu_1 \nu_2}\right)^2 & \geq \frac{N_1^2 K_1^2 N_2 K_2 M}{d^3 T^{2-\varepsilon}}, \\
 \left(\frac{K_1 K_2}{\nu_1 \nu_2}\right)^2 & \geq \frac{N_2 K_2 \mu_1}{N_1 \mu_2 \nu_2} T^\varepsilon = \frac{K_2 N_2 \mu_1}{d N_1} T^\varepsilon = \frac{K_2 N_2}{\nu_1 N_1} T^\varepsilon.
 \end{aligned}$$

Since

$$M \leq T^{\frac{1}{2}+\varepsilon} \left(\frac{N_2 K_2}{N_1 K_1}\right)^{\frac{1}{2}}$$

and

$$d \leq \frac{(N_1 K_1 N_2 K_2)^{\frac{1}{2}}}{T^{\frac{1}{2}+\eta}},$$

the first condition is implied by

$$T^{2-\varepsilon} \geq N_1^2 N_2^2 T^{\varepsilon-\eta},$$

which is true if $2\varepsilon < \eta$. The second condition is equivalent to

$$K_1^2 K_2 \geq \nu_1 \nu_2^2 T^\varepsilon \frac{N_2}{N_1}.$$

This is true as long as $\eta > \frac{1}{6}\xi_2 + \frac{4}{3}\varepsilon$, since

$$N_i \leq T^{\frac{1}{2}+\varepsilon}, \quad \frac{K_2}{K_1} \ll T^{\xi_2},$$

and

$$v_1 v_2^2 \frac{N_2}{N_1} T^\varepsilon \leq d^3 \frac{N_2}{N_1} T^\varepsilon \leq \frac{N_1^{\frac{1}{2}} N_2^{\frac{5}{2}} K_1^{\frac{3}{2}} K_2^{\frac{3}{2}} T^{\varepsilon-3\eta}}{T^{\frac{3}{2}}} \leq K_1^2 K_2 \frac{K_2^{\frac{1}{2}}}{K_1^{\frac{1}{2}}} T^{4\varepsilon-3\eta} \leq K_1^2 K_2 T^{4\varepsilon-3\eta+\frac{\xi_2}{2}}.$$

Applying Proposition 3 with

$$\delta^{-1} = \max(zC, yH) + 1 \ll T^\varepsilon$$

and using that

$$\mu_i, v_i \leq d \leq \frac{(N_1 K_1 N_2 K_2)^{\frac{1}{2}}}{T^{\frac{1}{2}+\eta}},$$

we obtain that (5.2) is bounded by

$$\begin{aligned} & \sum'_{M \leq T^{\frac{1}{2}+\varepsilon}} \sum_{d \leq Y} \frac{T^\varepsilon}{d} \sum_{\substack{\mu_1, v_1, \mu_2, v_2 \\ \mu_1 v_1 = \mu_2 v_2 = d}} \frac{dM}{N_2 K_2} \frac{dT}{MN_1 K_1} \frac{N_1 N_2 K_1 K_2}{Td^2} \frac{K_1}{v_1} \left(\frac{N_1}{\mu_1} + \frac{K_2}{v_2} T^{\frac{1}{2}} \right) \\ & \quad \times \left(1 + \frac{N_1 N_2 v_1 v_2}{Td^2} \right)^{\frac{1}{2}} \left(1 + \frac{dN_2}{\mu_2 K_1 N_1} \right)^{\frac{1}{2}} \left(1 + \frac{MN_1 N_2^2 v_1^3 v_2^2}{(K_1 K_2)^2 d T \mu_2^2} \right)^{\frac{1}{4}} \\ & \ll \sum'_{M \leq T^{\frac{1}{2}+\varepsilon}} \sum_{d \leq Y} \frac{T^\varepsilon}{d} \sum_{\substack{\mu_1, v_1, \mu_2, v_2 \\ \mu_1 v_1 = \mu_2 v_2 = d}} \frac{K_1}{v_1} \left(\frac{N_1}{\mu_1} + \frac{K_2}{v_2} T^{\frac{1}{2}} \right) \left(1 + \frac{dN_2}{\mu_2 K_1 N_1} \right)^{\frac{1}{2}} \\ & \quad \times \left(1 + \frac{MN_1 N_2^2 v_1^3 v_2^2}{(K_1 K_2)^2 d T \mu_2^2} \right)^{\frac{1}{4}} \\ & \ll \sum_{d \leq Y} \frac{T^\varepsilon}{d} \sum_{\substack{\mu_1, v_1, \mu_2, v_2 \\ \mu_1 v_1 = \mu_2 v_2 = d}} \left(\frac{K_1 K_2 T^{\frac{1}{2}}}{v_1 v_2} + \frac{K_1 N_1}{d} + \frac{K_1^{\frac{3}{8}} N_1^{\frac{9}{8}} N_2^{\frac{5}{8}}}{\mu_1^{\frac{3}{4}} \mu_2 K_2^{\frac{3}{8}} T^{\frac{1}{8}}} \right. \\ & \quad + \frac{K_1^{\frac{3}{8}} K_2^{\frac{5}{8}} T^{\frac{3}{8}} N_1^{\frac{1}{8}} N_2^{\frac{5}{8}}}{v_1^{\frac{1}{4}} d^{\frac{3}{4}}} + \frac{K_1^{\frac{1}{2}} K_2 T^{\frac{1}{2}} N_2^{\frac{1}{2}}}{v_1 v_2^{\frac{1}{2}} N_1^{\frac{1}{2}}} \\ & \quad \left. + \frac{K_1^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}}}{d^{\frac{1}{2}} \mu_2^{\frac{1}{2}}} + \frac{N_1^{\frac{5}{8}} N_2^{\frac{9}{8}} d^{\frac{1}{2}}}{\mu_1^{\frac{3}{4}} \mu_2^{\frac{3}{2}} K_2^{\frac{3}{8}} K_1^{\frac{1}{8}} T^{\frac{1}{8}}} + \frac{T^{\frac{3}{8}} N_2^{\frac{9}{8}} K_2^{\frac{5}{8}}}{N_1^{\frac{3}{8}} K_1^{\frac{1}{8}} d^{\frac{1}{4}} v_1^{\frac{1}{4}} \mu_2^{\frac{1}{2}}} \right) \\ & \ll K_1 K_2 T^{\frac{1}{2}+\varepsilon} + K_1 N_1 T^\varepsilon + \frac{K_1^{\frac{3}{8}} N_1^{\frac{9}{8}} N_2^{\frac{5}{8}}}{K_2^{\frac{3}{8}} T^{\frac{1}{8}-\varepsilon}} + K_1^{\frac{3}{8}} K_2^{\frac{5}{8}} T^{\frac{3}{8}+\varepsilon} N_1^{\frac{1}{8}} N_2^{\frac{5}{8}} \\ & \quad + \frac{K_1^{\frac{1}{2}} K_2 T^{\frac{1}{2}+\varepsilon} N_2^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} + K_1^{\frac{1}{2}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} T^\varepsilon + \frac{N_1^{\frac{7}{8}} N_2^{\frac{11}{8}} K_1^{\frac{1}{8}}}{K_2^{\frac{1}{8}} T^{\frac{3}{8}+\frac{\eta}{2}-\varepsilon}} + \frac{T^{\frac{3}{8}+\varepsilon} N_2^{\frac{9}{8}} K_2^{\frac{5}{8}}}{N_1^{\frac{3}{8}} K_1^{\frac{1}{8}}}. \end{aligned}$$

Summing over K_i and N_i , we obtain that (5.2) is bounded by

$$K^2 T^{\frac{1}{2}+\varepsilon} + T^{\frac{3}{4}+\frac{3\xi_2}{8}+\varepsilon} + K N^{\frac{3}{4}} T^{\frac{3}{8}+\varepsilon} + K^{\frac{3}{2}} T^{\frac{1}{2}+\varepsilon+\frac{\xi_1}{2}} + T^{\frac{3}{4}+\frac{\xi_2}{8}-\frac{\eta}{2}+\varepsilon} + K^{\frac{1}{2}} N^{\frac{3}{4}} T^{\frac{3}{8}+\frac{3\xi_1}{8}+\frac{\xi_2}{8}}.$$

Theorem 4 then follows by taking $\eta = \frac{\xi_2}{6} + 3\varepsilon$ and collecting the error term (5.1).

6. Proof of Corollary 2

The proof of Corollary 2 requires the following two lemmas.

Lemma 2. *Let G be a compactly supported function. If $F = -G'$ for $x > 0$ and F is three times continuously differentiable and compactly supported, then*

$$\sum_n \frac{G\left(\frac{\log n}{\log x}\right)}{n^s} = \frac{1}{2\pi i} \int_{(c)} \zeta(s+w) \cdot \widehat{F}\left(-\frac{iw \log x}{2\pi}\right) \frac{dw}{w}$$

for $c > \max(1 - \Re(s), 0)$ and $x > 1$, where \widehat{F} denotes the Fourier transform of F ,

$$\widehat{F}(x) := \int_{-\infty}^{\infty} F(u) e^{-2\pi i u x} du.$$

Proof. First of all $\widehat{F}(x)$ is entire because F is compactly supported. We expand the function $\zeta(s+w)$ into its Dirichlet series and compute

$$(6.1) \quad \frac{1}{2\pi i} \int_{(c)} n^{-w} \cdot \widehat{F}\left(-\frac{iw \log x}{2\pi}\right) \frac{dw}{w}.$$

Notice that

$$\widehat{F}\left(-\frac{iw \log x}{2\pi}\right) = \int_{-\infty}^{\infty} F(u) x^{uw} du.$$

Inserting this representation into (6.1) and inter-changing integrals, we obtain

$$\int_{-\infty}^{\infty} F(u) \cdot \frac{1}{2\pi i} \int_{(c)} \left(\frac{x^u}{n}\right)^w \frac{dw}{w} du = \int_{\frac{\log n}{\log x}}^{\infty} F(u) du = G\left(\frac{\log n}{\log x}\right).$$

In order to justify the interchange of the two integrations we truncate the integral in (6.1) at a large height X , committing an error which goes to zero as $X \rightarrow \infty$ (since the Fourier transform \widehat{F} will decay sufficiently fast), and interchange. Then, we use a Perron formula with error term in order to compute the conditionally convergent Perron integral appearing above. Taking the height $X \rightarrow \infty$ returns the desired result, as stated. \square

Lemma 3. *Let $A \geq 0$ and let $0 < \eta < \frac{1}{66}$ be fixed constants. Let $v \in \mathbb{R}$ and $x < T^{\frac{1}{2}+\eta}$. Let $s = \sigma + it$, where $\sigma = \frac{1}{2} + \frac{A}{\log T}$ and $T \leq t \leq 2T$. Then*

$$\int_T^{2T} |\zeta(s)|^2 \cdot \left| \sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}} \right|^2 dt \ll T(\log T)^{\frac{9}{4}},$$

where $d_{1/2}(n)$ are the coefficients of the Dirichlet series expansion

$$\zeta(s)^{\frac{1}{2}} = \sum_{n \geq 1} \frac{d_{1/2}(n)}{n^s}, \quad \Re(s) > 1.$$

Proof. Let

$$\Phi_{x,v}(s) := \zeta(s) \cdot \sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}}$$

and

$$f_t(s) = \frac{s-1}{s-3} \exp((s-it)^2).$$

Then, by Gabriel's convexity theorem (see [13, Lemma 3])

$$\begin{aligned} \int_{\mathbb{R}} |\Phi_{x,v}(\sigma + iu) f_t(\sigma + iu)|^2 du &\leq \left(\int_{\mathbb{R}} |\Phi_{x,v}(\tfrac{1}{2} + iu) f_t(\tfrac{1}{2} + iu)|^2 du \right)^{\frac{\frac{5}{2}-\sigma}{2}} \\ &\quad \times \left(\int_{\mathbb{R}} |\Phi_{x,v}(\tfrac{5}{2} + iu) f_t(\tfrac{5}{2} + iu)|^2 du \right)^{\frac{\sigma-\frac{1}{2}}{2}}. \end{aligned}$$

We now integrate both sides over $T \leq t \leq 2T$ and use Hölder's inequality to get

$$\begin{aligned} \int_{\mathbb{R}} |\Phi_{x,v}(\sigma + iu)|^2 \tilde{f}_T(\sigma + iu) du &\leq \left(\int_{\mathbb{R}} |\Phi_{x,v}(\tfrac{1}{2} + iu)|^2 \tilde{f}_T(\tfrac{1}{2} + iu) du \right)^{\frac{\frac{5}{2}-\sigma}{2}} \\ &\quad \times \left(\int_{\mathbb{R}} |\Phi_{x,v}(\tfrac{5}{2} + iu)|^2 \tilde{f}_T(\tfrac{5}{2} + iu) du \right)^{\frac{\sigma-\frac{1}{2}}{2}}, \end{aligned}$$

where

$$\tilde{f}_T(\sigma + iu) := \int_T^{2T} |f_t(\sigma + iu)|^2 dt.$$

Clearly $\tilde{f}_T(\sigma + iu) \asymp 1$ if $T \leq u \leq 2T$. In addition,

$$\tilde{f}_T(\sigma + iu) \ll \begin{cases} 1 & \text{if } \frac{T}{2} \leq u \leq 3T, \\ e^{-|u|} & \text{otherwise.} \end{cases}$$

We also note that $\Phi_{x,v}(s) \ll (1 + |s|)^{\frac{1}{4}+\varepsilon} \cdot \sqrt{T}$. Therefore the previous inequality becomes

$$\begin{aligned} \int_T^{2T} |\Phi_{x,v}(\sigma + it)|^2 dt &\ll \left(\int_{\frac{T}{2}}^{3T} |\Phi_{x,v}(\tfrac{1}{2} + it)|^2 dt + O(T) \right)^{\frac{\frac{5}{2}-\sigma}{2}} \\ &\quad \times \left(\int_{\frac{T}{2}}^{3T} |\Phi_{x,v}(\tfrac{5}{2} + it)|^2 dt + O(T) \right)^{\frac{\sigma-\frac{1}{2}}{2}}. \end{aligned}$$

According to Theorem 1 the first integral on the right-hand side is $O(T(\log T)^{\frac{9}{4}})$ while the second integral on the right-hand side is $O(T)$. \square

Let $\delta > 0$ be a small positive real number to be chosen later. We pick a parameter θ close to 1, with $\delta < \theta < 1$, and define

$$\widehat{F}(z) = e^{2\pi i(\theta-\delta)z} \cdot \left(\frac{e^{2\pi i(1-\theta)z} - 1}{2\pi i(1-\theta)z} \right)^N$$

with some bounded $N > 10$. We see that F is compactly supported on $[\theta-\delta, \theta-\delta+(1-\theta)N]$. Define for $x > 0$,

$$G(x) = 1 - \int_0^x F(u) du,$$

and $G(x) = 0$ for $x \leq -1$. Moreover, we let $G(x)$ decay smoothly until 0 on the interval $[-1, 0]$. This way $F = -G'$ for $x > 0$. We notice that

$$G(x) = \begin{cases} 1 & \text{for } 0 < x < \theta - \delta, \\ 1 - \widehat{F}(0) = 0 & \text{for } x > \theta - \delta + (1-\theta)N. \end{cases}$$

Finally, we notice that G is N times differentiable, and consequently that $\widehat{G}(x) \ll (1+|x|)^{-N}$.

Now we make a choice for θ and δ . Let

$$\theta = \frac{\log y}{\log x}$$

with $y = T^{\frac{1}{2}+2\delta}$ and x chosen so that $\theta - \delta + (1 - \theta)N < 1$. We pick $1 - \theta = (\delta/2)/(N - 1)$ so that $x = y^{1/(1-(\delta/2)/(N-1))}$. Then, we choose δ small enough but positive so as to ensure that

$$x < T^{\frac{1}{2} + \frac{1}{100}}.$$

Note that

$$\widehat{F}\left(-\frac{i w \log x}{2\pi}\right) = (yx^{-\delta})^w \cdot \left(\frac{(\frac{x}{y})^w - 1}{w(1 - \theta) \log x}\right)^N$$

Let $s = \sigma + it$ with $t \asymp T$ and $\sigma = \frac{1}{2} + \frac{A}{\log T}$, with $A > 0$. Using Lemma 2 and shifting contours to $\Re(w) = \frac{1}{2} - \sigma$, we get

$$\zeta(s) = \sum_n \frac{G\left(\frac{\log n}{\log x}\right)}{n^s} + \kappa(4N)^N \frac{(yx^{-\delta})^{\frac{1}{2}-\sigma}}{(\delta \log x)^N} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}} + O(T^{-1}),$$

where $O(T^{-1})$ is the contribution from the pole at $w = 1 - s$ and with $|\kappa| \leq 1$. Let

$$c(m) = \sum_{\substack{m=fe \\ f, e \leq x}} d_{1/2}(e) d_{1/2}(f).$$

Importantly, notice that $c(m) = 1$ for $m \leq x$. Since in addition $G(v) = 0$ for $v > 1$, we get

$$\begin{aligned} \sum_n \frac{G\left(\frac{\log n}{\log x}\right)}{n^s} &= \sum_n c(n) \frac{G\left(\frac{\log n}{\log x}\right)}{n^s} \\ &= \frac{\log x}{2\pi} \int_{\mathbb{R}} \sum_n \frac{c(n)}{n^{s+iv}} \cdot \widehat{G}\left(\frac{v \log x}{2\pi}\right) dv \\ &= \frac{\log x}{2\pi} \int_{\mathbb{R}} \left(\sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}} \right)^2 \widehat{G}\left(\frac{v \log x}{2\pi}\right) dv. \end{aligned}$$

Combining the above two equations, we have obtained the following inequality:

$$\begin{aligned} |\zeta(s)| &\leq \log x \int_{\mathbb{R}} \left| \sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}} \right|^2 \cdot \left| \widehat{G}\left(\frac{v \log x}{2\pi}\right) \right| dv \\ &\quad + (4N)^N \frac{(yx^{-\delta})^{\frac{1}{2}-\sigma}}{(\delta \log x)^N} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}} + O\left(\frac{1}{T}\right). \end{aligned}$$

Therefore we have obtained

$$\int_T^{2T} |\zeta(s)|^3 dt \leq \log x \int_{\mathbb{R}} \left| \widehat{G}\left(\frac{v \log x}{2\pi}\right) \right| \int_T^{2T} |\zeta(s)|^2 \cdot \left| \sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}} \right|^2 dt dv + \mathcal{E} + O(T^\varepsilon),$$

where

$$\begin{aligned} (6.2) \quad \mathcal{E} &\leq (4N)^N \cdot \frac{(yx^{-\delta})^{\frac{1}{2}-\sigma}}{(\delta \log x)^N} \int_{-\infty}^{\infty} \left(\int_T^{2T} |\zeta(s)|^2 \cdot |\zeta(\frac{1}{2} + it + iv)| dt \right) \\ &\quad \times \frac{dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}}. \end{aligned}$$

By Hölder's inequality and the bound $|\zeta(\frac{1}{2} + it)| \ll (1 + |t|)^{\frac{1}{6} + \varepsilon}$, for $|v| < T^{\frac{1}{100}}$ we have

$$\begin{aligned} \int_T^{2T} |\zeta(s)|^2 |\zeta(\frac{1}{2} + it + iv)| dt &\leq \left(\int_T^{2T} |\zeta(s)|^3 dt \right)^{\frac{2}{3}} \left(\int_T^{2T} |\zeta(\frac{1}{2} + it + iv)|^3 dt \right)^{\frac{1}{3}} \\ &\leq M_3(\sigma, T)^{\frac{2}{3}} \cdot \left(M_3(\frac{1}{2}, T) + O(|v|T^{\frac{1}{2} + \varepsilon}) \right)^{\frac{1}{3}}, \end{aligned}$$

where

$$M_3(\sigma, T) := \int_T^{2T} |\zeta(\sigma + it)|^3 dt.$$

By a minor modification of [13, Lemma 4] we have

$$M_3(\frac{1}{2}, T) \leq C \cdot T^{\frac{3}{2}(\sigma - \frac{1}{2})} M_3(\sigma, T).$$

Therefore,

$$\begin{aligned} \int_T^{2T} |\zeta(s)|^2 |\zeta(\frac{1}{2} + it + iv)| dt &\leq M_3(\sigma, T)^{\frac{2}{3}} \cdot \left(C T^{\frac{3}{2}(\sigma - \frac{1}{2})} M_3(\sigma, T) + O(|v|T^{\frac{1}{2} + \varepsilon}) \right)^{\frac{1}{3}} \\ &\leq C T^{\frac{1}{2}(\sigma - \frac{1}{2})} \cdot (M_3(\sigma, T) + O(T^{1 - \varepsilon})). \end{aligned}$$

The contribution of $|v| > T^{\frac{1}{100}}$ to (6.2) is negligible, provided that N is chosen to be large enough. We conclude that

$$\mathcal{E} \leq \frac{C(4N)^N}{\delta^N} \left(\frac{T^{\frac{1}{2}} x^\delta}{y} \right)^{\sigma - \frac{1}{2}} \cdot \frac{M_3(\sigma, T)}{(\log x(\sigma - \frac{1}{2}))^N} + O_A(T^{1 - \varepsilon}).$$

Using Lemma 3, we find that

$$\log x \int_{\mathbb{R}} \left| \widehat{G}\left(\frac{v \log x}{2\pi}\right) \right| \int_T^{2T} |\zeta(s)|^2 \cdot \left| \sum_{n \leq x} \frac{d_{1/2}(n)}{n^{s+iv}} \right|^2 dt dv \ll T(\log T)^{\frac{9}{4}}.$$

We have obtained the inequality

$$M_3(\sigma, T) \ll T(\log T)^{\frac{9}{4}} + O_A(T^{1 - \varepsilon}) + \frac{C(4N)^N}{\delta^N} \cdot \left(\frac{T^{\frac{1}{2}} x^\delta}{y} \right)^{\sigma - \frac{1}{2}} \cdot \frac{M_3(\sigma, T)}{((\sigma - \frac{1}{2}) \log x)^N}.$$

Recall that $y = T^{\frac{1}{2} + 2\delta} < x < T$. Since $\sigma = \frac{1}{2} + \frac{A}{\log T}$, the third term on the right-hand side in the above equation is less than

$$\leq C \left(\frac{8N}{A} \right)^N \delta^{-N} e^{-\delta A} M_3(\sigma, T)$$

with C an absolute constant. Thus, if A is large enough (but bounded), then the third term on the right-hand side in the above equation is absorbed into the left-hand side, and we conclude that

$$M_3(\sigma, T) \ll T(\log T)^{\frac{9}{4}}.$$

Since

$$M_3(\frac{1}{2}, T) \ll T^{\frac{3}{2}(\sigma - \frac{1}{2})} M_3(\sigma, T)$$

by Heath-Brown [13, Lemma 4] and since $\sigma = \frac{1}{2} + \frac{A}{\log T}$, we obtain that

$$M_3(\frac{1}{2}, T) \ll T(\log T)^{\frac{9}{4}}.$$

6.1. Moments of the form $k = 1 + \frac{1}{n}$. Since we do not claim the result for moments with $k = 1 + \frac{1}{n}$ we only sketch the necessary modifications of the previous argument, for the convenience of the interested reader. In order to adapt our argument above to moments of the form $1 + \frac{1}{n}$, it suffices to prove the inequality

$$|\zeta(s)|^{\frac{2}{n}} \ll \log T \int_{\sigma_-}^{\sigma_+} \int_{\mathbb{R}} \left| \hat{G}\left(\frac{v}{2\pi}\right) \right|^{\frac{2}{n}} \cdot \left| \sum_{m \leq x} \frac{d_{1/n}(m)}{m^{s+\sigma+\frac{iv}{\log x}}} \right|^2 dv d\sigma \\ + \frac{(yx^{-\delta})^{\frac{2}{n}(\frac{1}{2}-\sigma)}}{(\delta \log x)^{\frac{2N}{n}}} \cdot \log T \int_{\sigma_-}^{\sigma_+} \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it + iv)|^{\frac{2}{n}} dv d\sigma}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{n}}},$$

where

$$\sigma_- = \frac{1}{2} - \frac{1}{\log T}, \quad \sigma_+ = \frac{1}{2} + \frac{1}{\log T}$$

and with the implicit constant depending at most on n, N and the same choice of parameters θ, x, y, δ . This is sufficient because the previous argument does not depend on some specific quantification of the dependence on N . First we note that

$$\sum_n \frac{G(\frac{\log n}{\log x})}{n^s} = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\sum_{m \leq x} \frac{d_{1/n}(m)}{m^{s+\frac{iv}{\log x}}} \right)^n \cdot \hat{G}\left(\frac{v}{2\pi}\right) dv.$$

Combining this with Lemma 2, and using the same choice of parameters θ, x, y, δ as before, it follows that

$$|\zeta(s)| \ll \int_{\mathbb{R}} \left| \hat{G}\left(\frac{v}{2\pi}\right) \right| \cdot \left| \sum_{m \leq x} \frac{d_{1/n}(m)}{m^{s+\sigma+\frac{iv}{\log x}}} \right|^n dv + \frac{(yx^{-\delta})^{\frac{1}{2}-\sigma}}{(\delta \log x)^N} \int_{\mathbb{R}} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}}.$$

Taking the $\frac{2}{n}$ power on both sides, it remains to show that

$$\left(\int_{\mathbb{R}} \left| \hat{G}\left(\frac{v}{2\pi}\right) \right| \cdot \left| \sum_{m \leq x} \frac{d_{1/n}(m)}{m^{s+\frac{iv}{\log x}}} \right|^n dv \right)^{\frac{2}{n}} \\ \ll \log T \int_{\sigma_-}^{\sigma_+} \int_{\mathbb{R}} \left| \hat{G}\left(\frac{v}{2\pi}\right) \right|^{\frac{2}{n}} \cdot \left| \sum_{m \leq x} \frac{d_{1/n}(m)}{m^{s+\sigma+\frac{iv}{\log x}}} \right|^2 dv d\sigma,$$

and that

$$(6.3) \quad \left(\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}} \right)^{\frac{2}{n}} \\ \ll \log T \int_{\sigma_-}^{\sigma_+} \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it + iv)|^{\frac{2}{n}} dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{n}}} d\sigma.$$

We will only show how to prove the second inequality since the proof of the first is very similar. We bound the integral

$$\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}} \leq (\sigma - \frac{1}{2})^{-N-1} \sum_k \frac{M_k}{(1 + |k|)^{N+1}},$$

where M_k is the maximum of $|\zeta(\frac{1}{2} + it + iv)|$ over the interval $|v - k(\sigma - \frac{1}{2})| < \frac{1}{2}(\sigma - \frac{1}{2})$.

Therefore

$$(6.4) \quad \left(\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + it + iv)| dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{2}}} \right)^{\frac{2}{n}} \leq (\sigma - \frac{1}{2})^{-\frac{2(N+1)}{n}} \sum_k \frac{M_k^{\frac{2}{n}}}{(1 + |k|)^{\frac{2(N+1)}{n}}}.$$

By sub-harmonicity,

$$M_k^{\frac{2}{n}} \ll \log T \int_{\sigma_-}^{\sigma_+} \int_{(k-1)(\sigma-\frac{1}{2})}^{(k+1)(\sigma-\frac{1}{2})} |\zeta(\sigma + it + ix)|^{\frac{2}{n}} dx d\sigma.$$

We conclude that

$$\begin{aligned} & (\sigma - \frac{1}{2})^{-\frac{2(N+1)}{n}} \sum_k \frac{M_k^{\frac{2}{n}}}{(1 + |k|)^{\frac{2(N+1)}{n}}} \\ & \ll \log T \int_{\sigma_-}^{\sigma_+} \sum_k \int_{(k-1)(\sigma-\frac{1}{2})}^{(k+1)(\sigma-\frac{1}{2})} \frac{|\zeta(\sigma + it + ix)|^{\frac{2}{n}} dx d\sigma}{((\sigma - \frac{1}{2})^2 + (k(\sigma - \frac{1}{2}))^2)^{\frac{2(N+1)}{n}}} \\ & \ll \log T \int_{\sigma_-}^{\sigma_+} \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it + iv)|^{\frac{2}{n}} dv}{((\sigma - \frac{1}{2})^2 + v^2)^{\frac{N+1}{n}}} d\sigma. \end{aligned}$$

Combining these equations together, we obtain the desired inequality (6.3).

A. On Conjecture 1

Proposition 4. *Let $A, M, N \geq 1$ and let $A \ll (MN)^{\frac{1}{2}+\varepsilon}$. Then*

$$\max_{\alpha, \beta, v} |S_{A, M, N}| \gg (AMN)^{\frac{1}{2}-\varepsilon} (M + N)^{\frac{1}{2}} + A(M + N)^{1-\varepsilon}$$

for all $\varepsilon > 0$, where the maximum is taken over all choices of coefficients $\alpha_m, \beta_n, v_a \ll 1$.

Proof. By the reciprocity relation $\frac{\overline{m}}{n} \equiv -\frac{\overline{n}}{m} + \frac{1}{mn} \pmod{1}$, we can assume $M \geq N$. Moreover, we can assume $N, A \gg M^\varepsilon$ for some small $\varepsilon > 0$ and M arbitrary large, since otherwise the result is easy.

First, we consider the case $M^{1-\delta} \gg N$ for some $\delta > 0$ and we take $\alpha_m = f(m)$ for some smooth function $f : [M, 2M] \rightarrow [0, 1]$ which is such that $f^{(j)}(x) \ll_j x^{-j}$ for all $j \geq 0$ and $\int_{\mathbb{R}} f(x) dx = KM$, for some $K > 0$. Also, let $\beta_n = -\gamma_n$, where γ_n is the indicator function of the primes congruent to 1 (mod 4) in $[N, 2N]$, and let v_a be the indicator function of the primes congruent to 3 (mod 4) in $[A, 2A]$.

By Poisson summation, we have

$$\sum_m f(m) e\left(\frac{a\overline{m}}{n}\right) = K \frac{M}{N} (c_n(a) + O(M^{-100})),$$

where

$$c_n(a) = \sum_{\substack{b=1 \\ (b,n)=1}}^n e\left(\frac{ba}{n}\right) = \mu\left(\frac{n}{(n,a)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,a)}\right)}$$

is the Ramanujan sum. It follows that

$$\begin{aligned} S_{A,M,N} &= K \frac{M}{N} \sum_a \sum_n \beta_n \nu_a (c_n(a) + O(M^{-100})) \\ &= K \frac{M}{N} \sum_a \sum_n \gamma_n \nu_a (1 + O(M^{-100})) \gg (MA)^{1-\varepsilon}. \end{aligned}$$

We now prove

$$\max_{\alpha, \beta, \nu} |S_{A,M,N}| \gg M(AN)^{\frac{1}{2}-\varepsilon},$$

which then implies the proposition even in the case $M^{1-\delta} \ll N$ for all $\delta > 0$.

By choosing α_m appropriately, we have

$$\max_{\alpha, \beta, \nu} |S_{A,M,N}| \gg \max_{\beta, \nu} \sum_m F_{m;\beta, \nu},$$

where

$$F_{m;\beta, \nu} := \left| \sum_a \sum_{(n,m)=1} \nu_a \beta_n e\left(\frac{a\bar{m}}{n}\right) \right|.$$

First, notice that we have

$$\max_{\beta, \nu} \sum_m F_{m;\beta, \nu} \geq \frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)}$$

with q any prime greater than $4(A^4 + N^4)$ and where $\beta(\chi_1), \nu(\chi_2)$ denotes sequences defined by $\beta(\chi_1)_n = \chi_1(n)$ and $\nu(\chi_2)_a = \chi_2(a)$ respectively. Moreover, by Hölder's inequality,

$$\begin{aligned} &\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)}^2 \\ &\leq \left(\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)} \right)^{\frac{2}{3}} \left(\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)}^4 \right)^{\frac{1}{3}}. \end{aligned}$$

The left-hand side is

$$\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)}^2 = \sum_m \sum_a \sum_{(n,m)=1} 1 \gg MAN,$$

and we also have

$$\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} \sum_m F_{m;\beta(\chi_1), \nu(\chi_2)}^4 = \sum_m \sum_{a_1 a_2 = a_3 a_4} \sum_{\substack{n_1 n_2 = n_3 n_4 \\ (m, n_1 n_2) = 1}} 1 \ll M(AN)^{2+\varepsilon}.$$

Thus,

$$\frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2 \pmod{q}} F_{m;\beta(\chi_1), \nu(\chi_2)} \gg M(AN)^{\frac{1}{2}-\varepsilon}$$

and the proposition follows. \square

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